MATHEMATICAL DESCRIPTIONS OF ELECTROMAGNETIC WAVES

LECTURE 4

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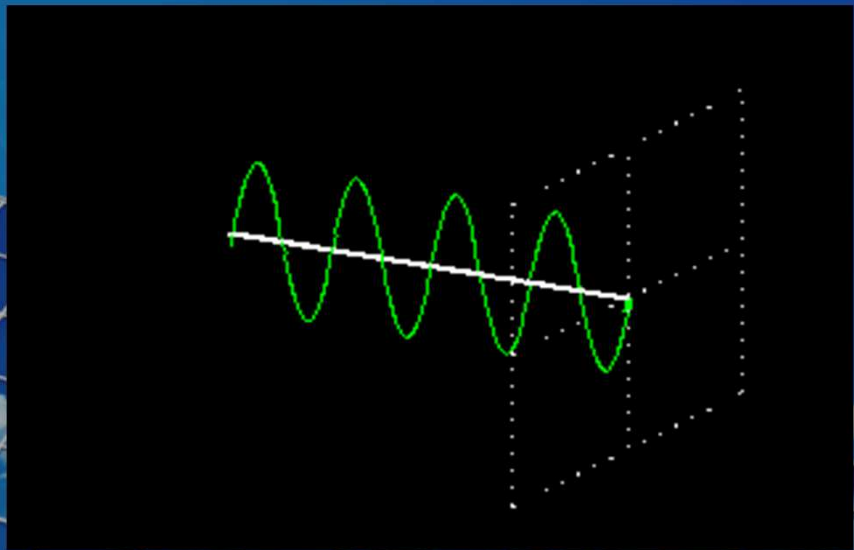
MATHEMATICAL FORMS OF ELECTROMAGENTIC WAVE

- **I. Basic concepts: Electromagnetic waves and types of polarization**
  - Plane-polarized wave: Horizontal
  - Plane-polarized wave: Vertical
  - Superposition of plane-polarized waves: Horizontal + Vertical → 45º Plane
  - Superposition of plane-polarized waves: Horizontal + Vertical → Right circular
  - Superposition of plane-polarized waves: Horizontal + Vertical → Left circular
  - Circularly polarized waves: Right and Left
  - Superposition of circularly polarized waves: Right + Left circular → Plane!
- **II. Interaction of light and matter**
  - Plane-polarized wave: Absorption
  - Circularly polarized wave: Absorption
  - Plane-polarized wave: Refraction
  - Circularly polarized wave: Refraction
  - Circular dichroism
  - Circular birefringence
  - Circular dichroism AND birefringence

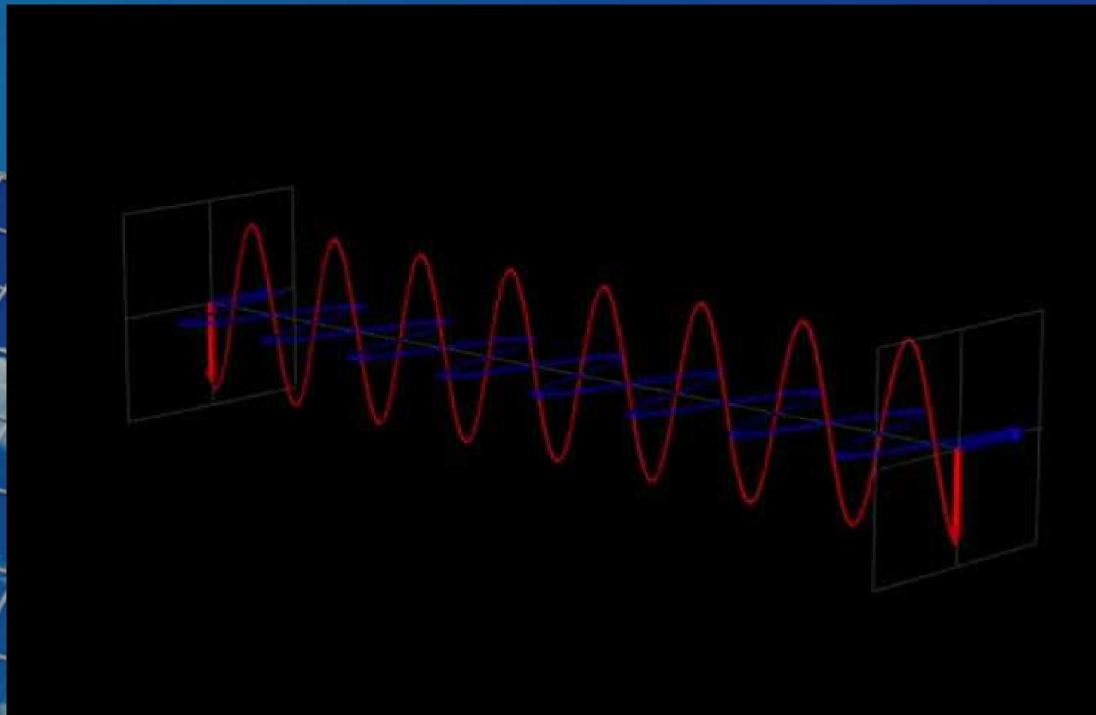
MATHEMATICAL CONCEPT
PLANE EM WAVE



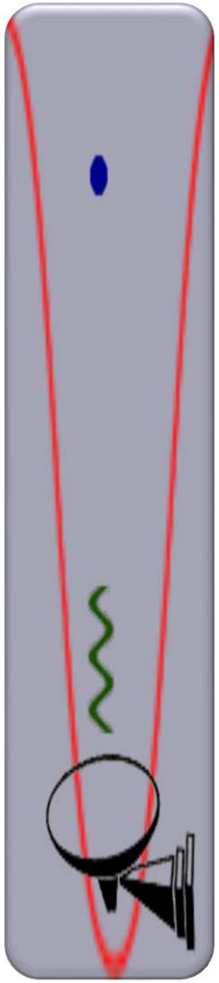
$$E_y = A\sin(x/\lambda - \omega t)$$

Vertically (y axis) polarized wave having an amplitude *A*, a wavelength of λ, and an angular velocity (frequency * 2π) of ω, propagating along

ANOTHER LOOK



Explanation of Frequency



Everything you see, and can't see, resonate at a specific frequency.

Frequencies are sinusoidal waves.

Speed of Light = Frequency x Wavelength

Example: The wavelength of a signal resonating at 3kHz is:

$$\frac{3 \times 10^{8}\ \text{m/s}}{3 \times 10^{3}\ \text{Hz}} = 100 \text{ kilometers or} \approx 62 \text{ miles!}$$

Lower frequencies have longer wavelengths. This characteristic allows these frequencies to be used for Morse code and amateur radio.

Phase (waves)

The phase of an oscillation or wave refers to a sinusoidal function such as the following:

$$x(t) = A \cdot \cos(2\pi f t + \theta)$$

$$y(t) = A \cdot \sin(2\pi f t + \theta) = A \cdot \cos(2\pi f t + \theta - \pi/2)$$

The term phase can refer to several different things:

- It can refer to a specified reference, such as $\cos(2\pi ft)$, in which case we would say the **phase** of $x(t)$ is θ, and the **phase** of $y(t)$ is $\theta-\pi/2$.
- It can refer to θ, in which case we would say $x(t)$ and $y(t)$ have the same **phase** but are relative to different references.
- In the context of communication waveforms, the time-variant angle $2\pi ft+\theta$, or its modulo 2π value, is referred to as instantaneous phase, but often just **phase**. *Instantaneous phase* has a formal definition that is applicable to more general functions and unambiguously defines a function's initial **phase** at t=0. Accordingly, it is θ for $x(t)$ and $\theta-\pi/2$ for $y(t)$. (also see phasor)

Instantaneous phase

In signal processing, the **instantaneous phase** (or "local phase" or simply "phase") of a complex-valued function x(t) is the real-valued function:

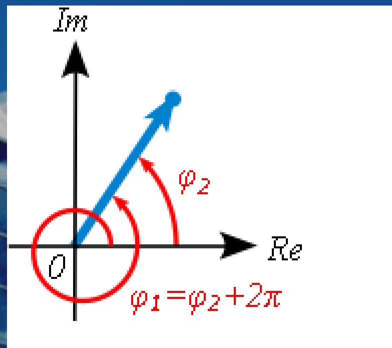
- Geometrically, in relation to an Argand diagram, $\arg z$ is the angle φ from the positive real axis to the vector representing z. The numeric value is given by the angle in radians and is positive if measured anticlockwise.
- Algebraically, an argument of the complex number $z = x + iy$ is any real quantity φ such that

$$z = x + iy = r\cos\varphi + ir\sin\varphi$$

for some positive real r. The quantity r is the *modulus* of z, written

$$r = |z| = \sqrt{x^2 + y^2}.$$

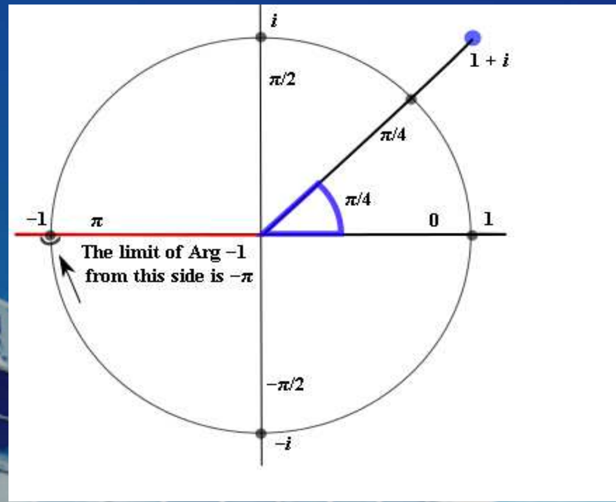
The names **amplitude**[1] or **phase**[2] are sometimes used equivalently.



Principal value

Open-closed interval

[-π, π]



Some authors define the range of the principal value as being in the closed-open interval $[0, 2\pi)$.

The set of all possible values of the argument can be written in terms of Arg as:

$$\arg z = \{\operatorname{Arg} z + 2\pi n : n \in \mathbb{Z}\}.$$

$$z = |z|\, e^{i \arg(z)}.$$

This is only really valid if z is non-zero but can be considered as valid also for $z = 0$ if $\arg(0)$ is considered as being an indeterminate form rather than as being undefined.

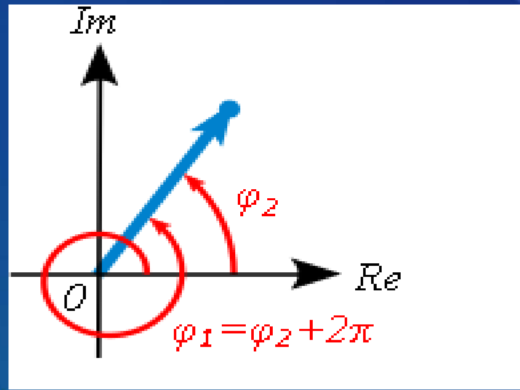
Some further identities follow. If z_1 and z_2 are two non-zero complex numbers then

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi}, \text{ and}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \pmod{2\pi}.$$

If $z \neq 0$ and n is any integer then

$$\arg(z^n) = n \arg(z) \pmod{2\pi}.$$



When $\phi(t)$ is constrained to an interval such as $(-\pi,\pi]$ or $[0,2\pi)$, it is called the **wrapped** phase. Otherwise it is called **unwrapped**, which is a continuous function of argument t, assuming s_a is a continuous function of t. Unless otherwise indicated, the continuous form should be inferred.

Examples

Example 1: $s(t) = A \cdot \cos(\omega t + \theta)$, where A and ω are positive values.

$$s_a(t) = A \cdot e^{i(\omega t + \theta)}$$

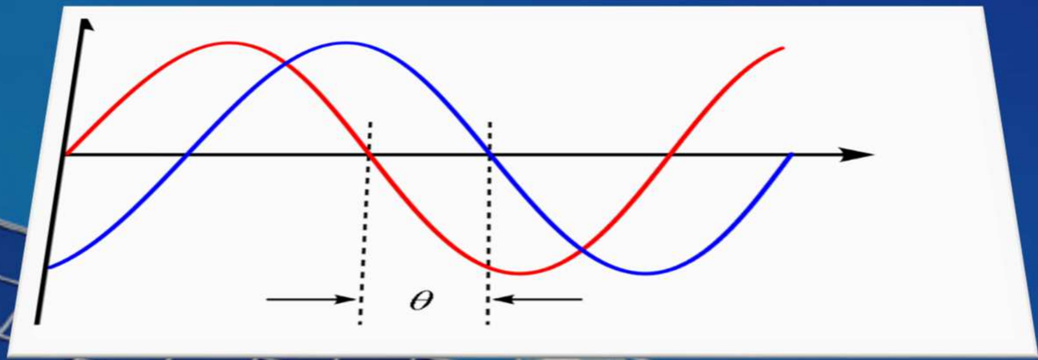
$$\phi(t) = \omega t + \theta$$

Example 2: $s(t) = A \cdot \sin(\omega t) = A \cdot \cos\left(\omega t - \frac{\pi}{2}\right)$

$$s_a(t) = A \cdot e^{i\left(\omega t - \frac{\pi}{2}\right)}$$

$$\phi(t) = \omega t - \frac{\pi}{2}$$

Phase shift



The horizontal axis represents an angle (phase) that is increasing with time

Phase shift

θ is sometimes referred to as a phase-shift, because it represents a "shift" from zero phase. But a change in θ is also referred to as a phase shift.

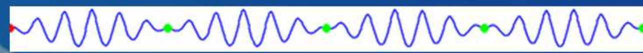
For infinitely long sinusoids, a change in θ is the same as a shift in time, such as a time-delay. If $x(t)$ is delayed (time-shifted) by $\frac{1}{4}$ of its cycle, it becomes:

$$\begin{aligned} x(t - \tfrac{1}{4}T) &= A \cdot \cos(2\pi f(t - \tfrac{1}{4}T) + \theta) \\ &= A \cdot \cos(2\pi f t - \tfrac{\pi}{2} + \theta), \end{aligned}$$

whose "phase" is now $\theta - \frac{\pi}{2}$. It has been shifted by $\frac{\pi}{2}$ radians.

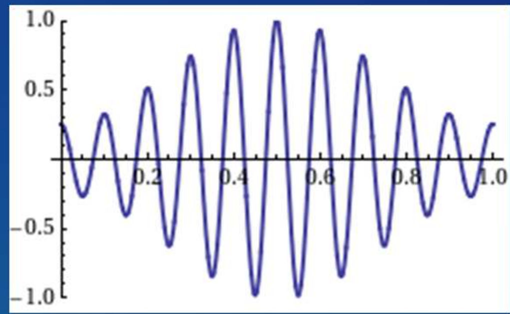
Phase velocity

The **phase velocity** of a wave is the rate at which the phase of the wave propagates in space. This is the speed at which the phase of any one frequency component of the wave travels. For such a component, any given phase of the wave (for example, the crest) will appear to travel at the phase velocity. The phase velocity is given in terms of the wavelength λ (lambda) and period T as



$$v_p = \frac{\lambda}{T}.$$

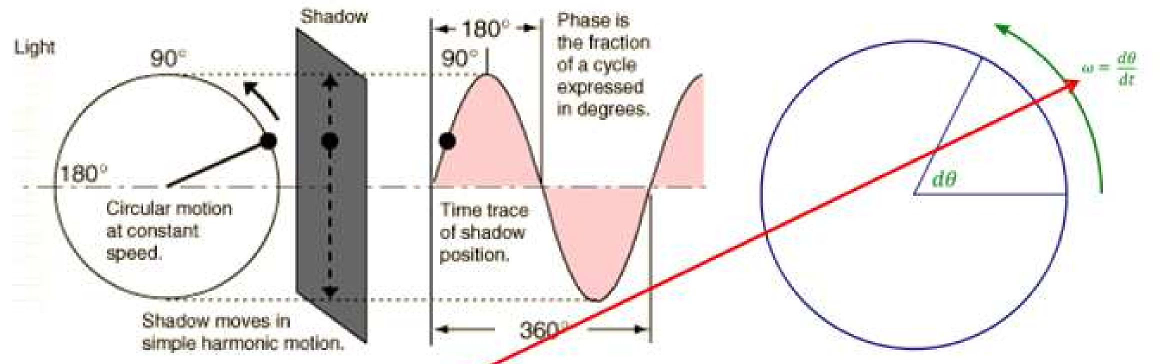
Angular Frequency



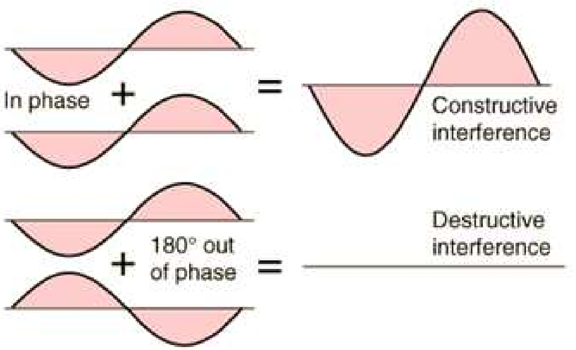
angular frequency ω, which is amount of oscillations per unit of time, and wavenumber k, which is amount of oscillations per unit of space, by

$$v_{\mathrm{p}} = \frac{\omega}{k}.$$

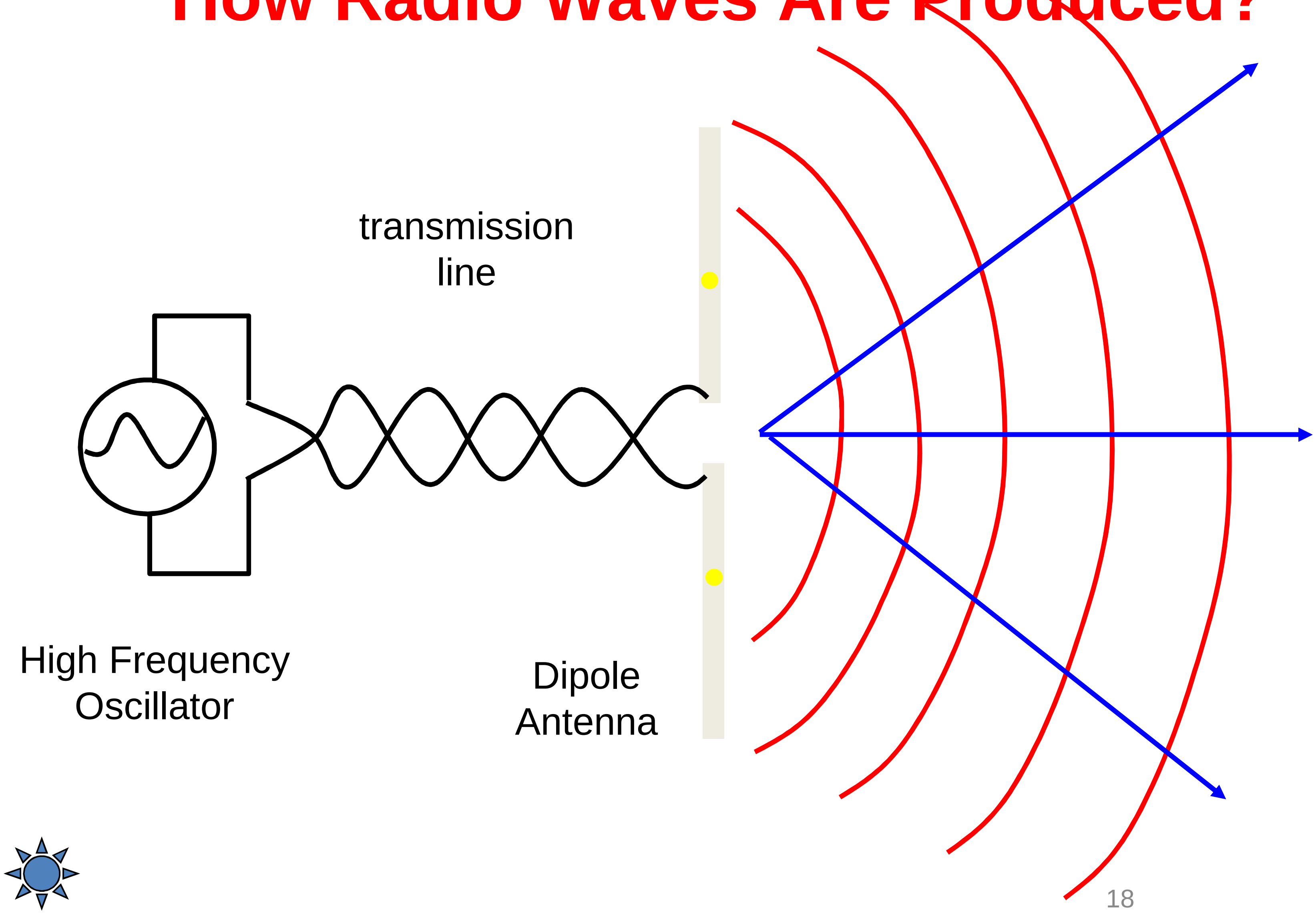
Wave phase and angular frequency



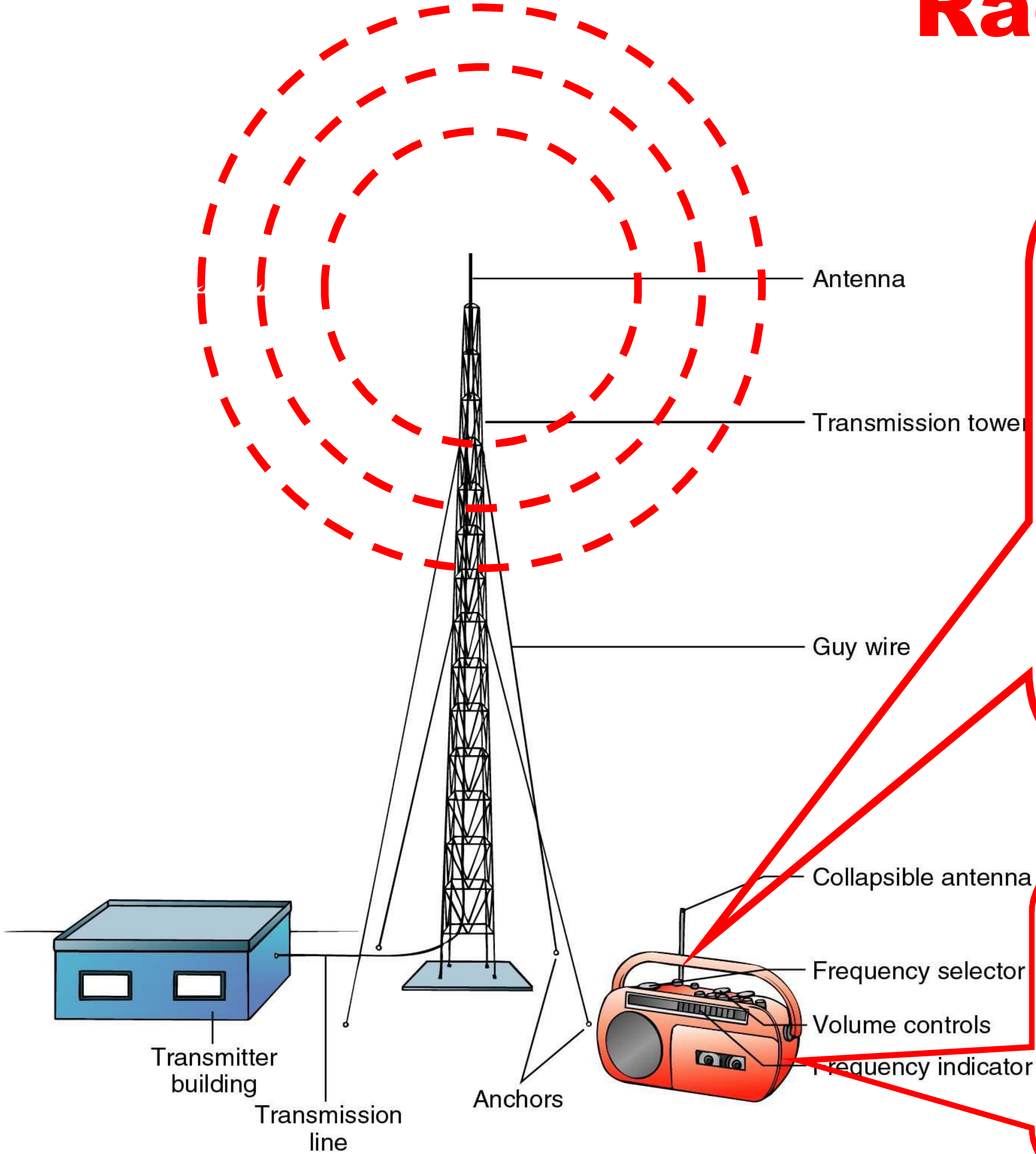
- Angular frequency $\omega = 2\pi f = 2\pi/T$
- Frequency with which phase changes
- Angles in radians (rad)
  - $360° = 2\pi$ rad, so 1 rad $= 360/2\pi = 57.3°$
  - Rad to deg. ($*180/\pi$) and deg. to rad ($*\pi/180$)



How Radio Waves Are Produced?



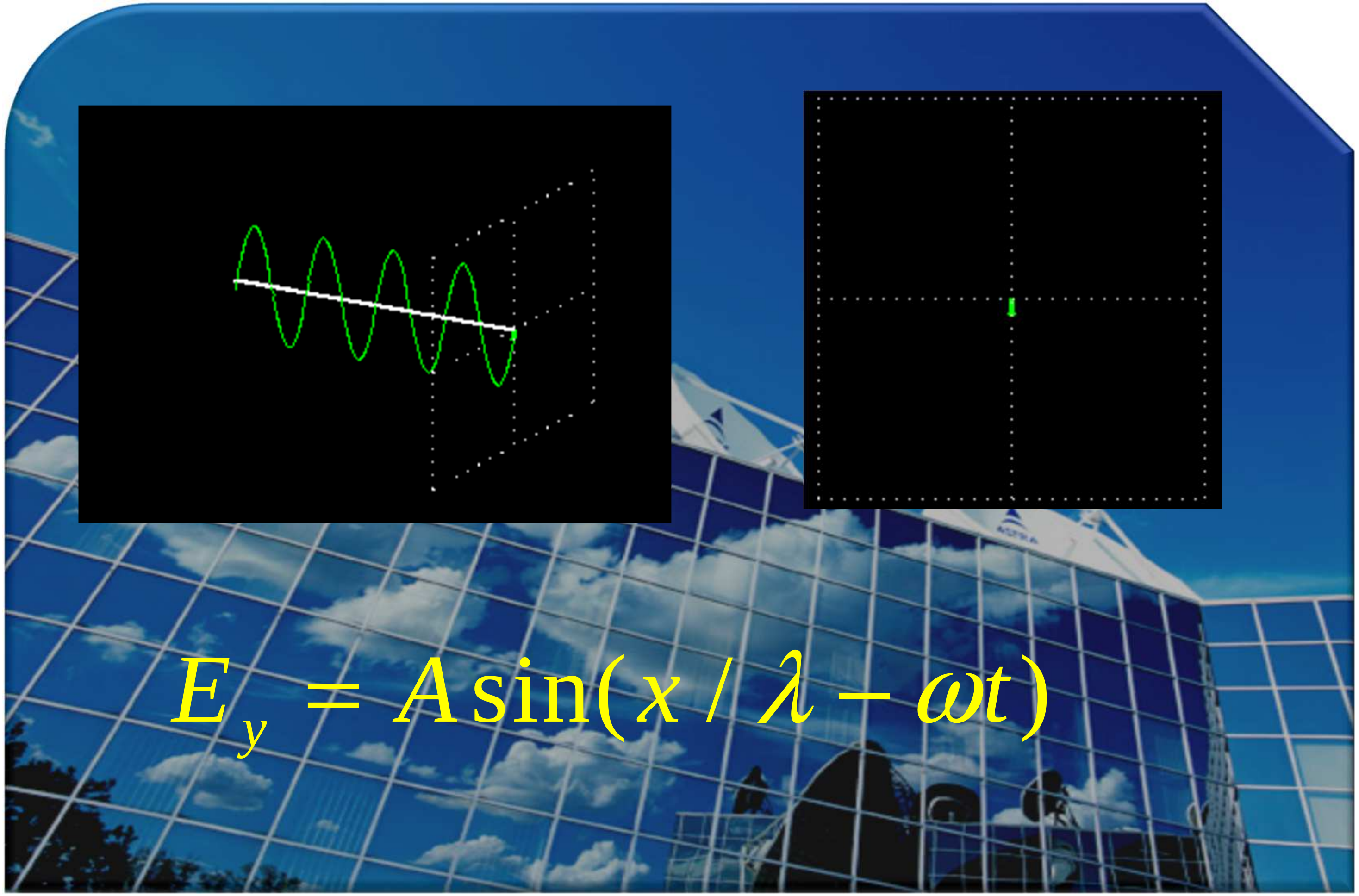
Radio Antenna



the oscillating electric field of the EM wave causes the electrons in the receiving antenna to oscillate at the same frequency

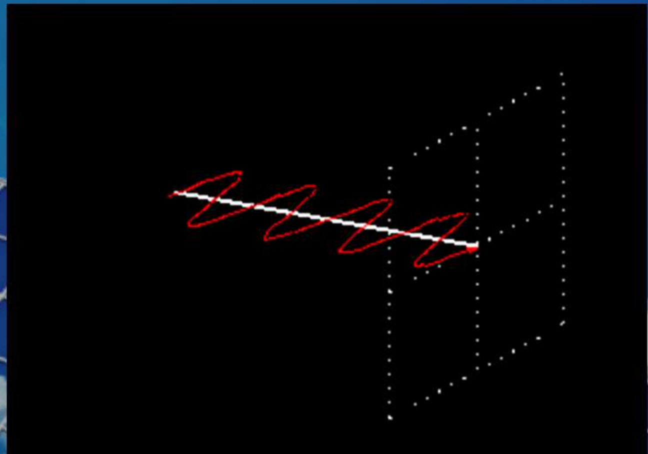
the amplifier converts the electrical signal to sound waves

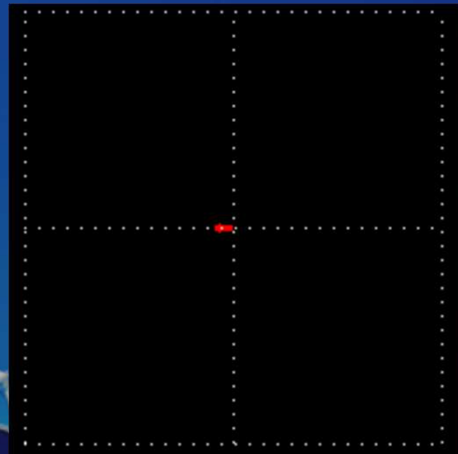
Plane-polarized EM



$$E_y = A\sin(x/\lambda - \omega t)$$

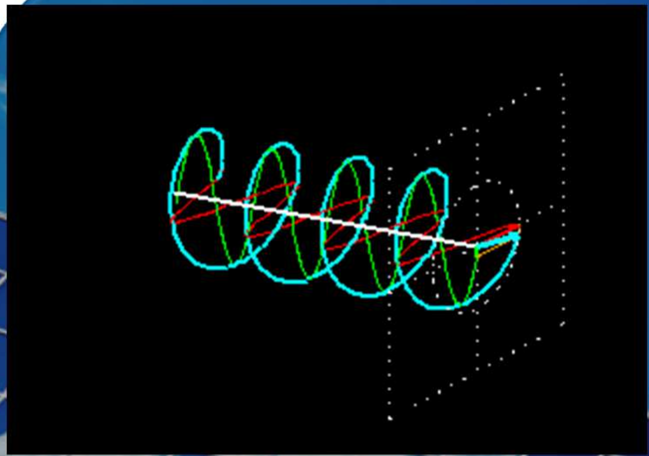
Horizontal

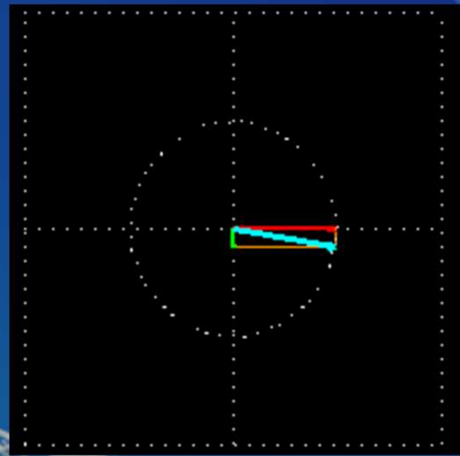




$$E_z = A\sin(x / \lambda - \omega t)$$

Circularly polarized EM

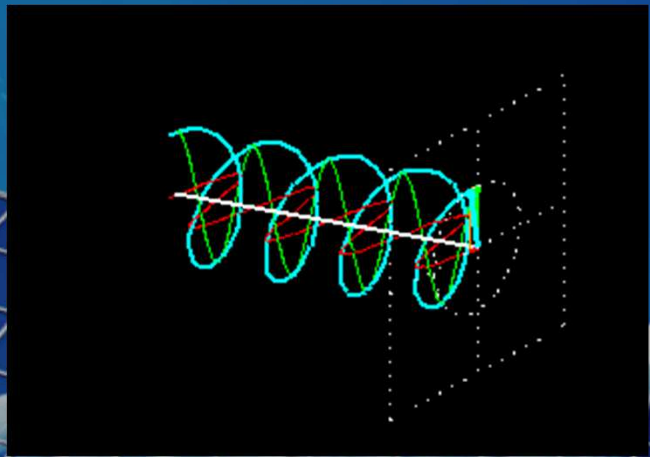


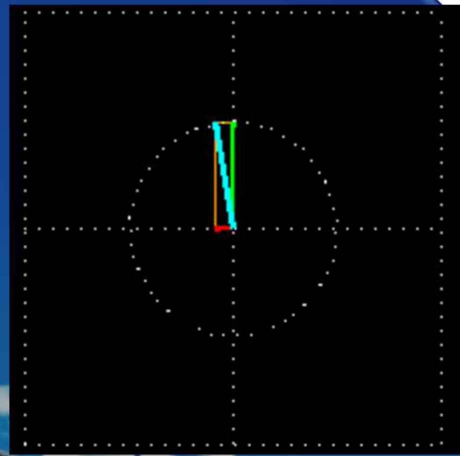


$$E_y = A\sin(x/\lambda - \omega t + 90^\circ)$$

$$E_z = A\sin(x/\lambda - \omega t)$$

Circularly polarized EM

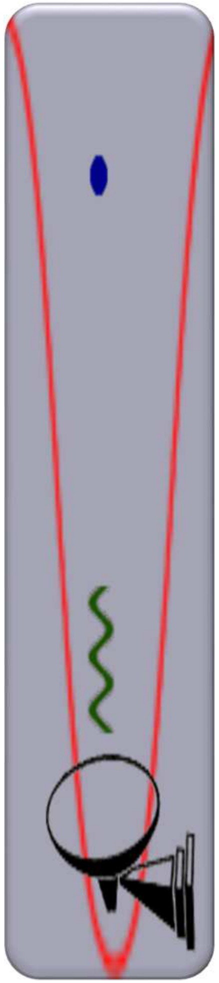




$$E_y = A\sin(x/\lambda - \omega t + 90^\circ)$$

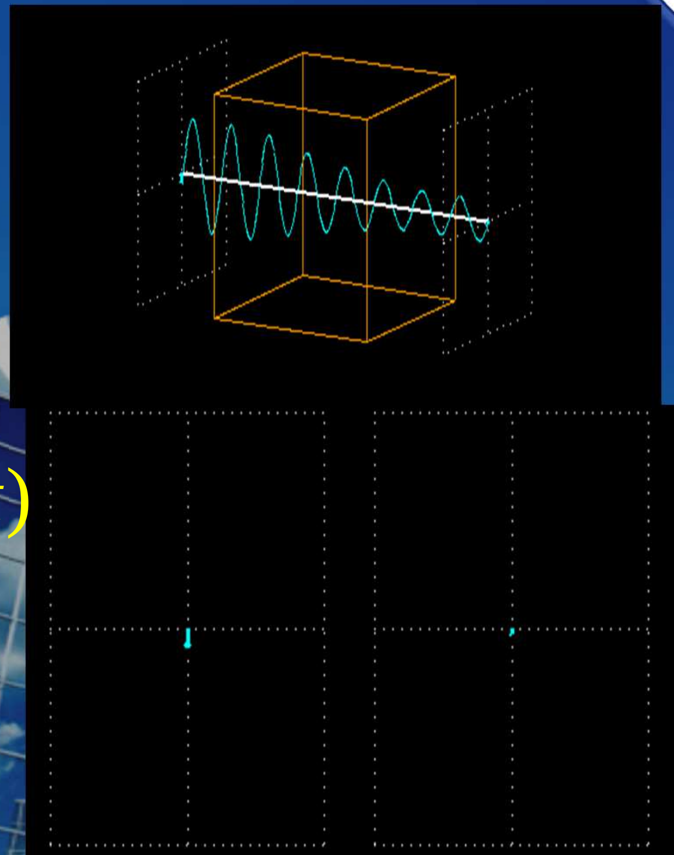
$$E_z = A\sin(x/\lambda - \omega t)$$

Interaction of EM and matter: Absorption



Material with an *extinction coefficient* ε

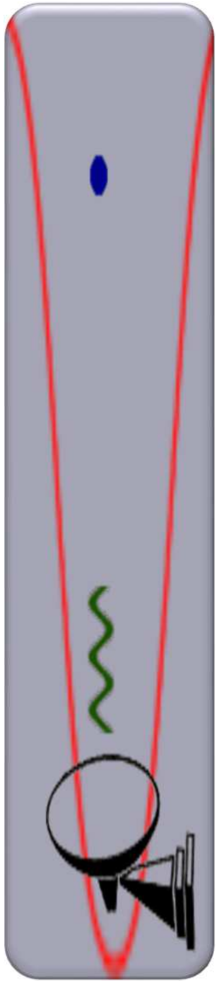
$$E_y = Ae^{-\varepsilon x}\sin(x/\lambda - \omega t)$$

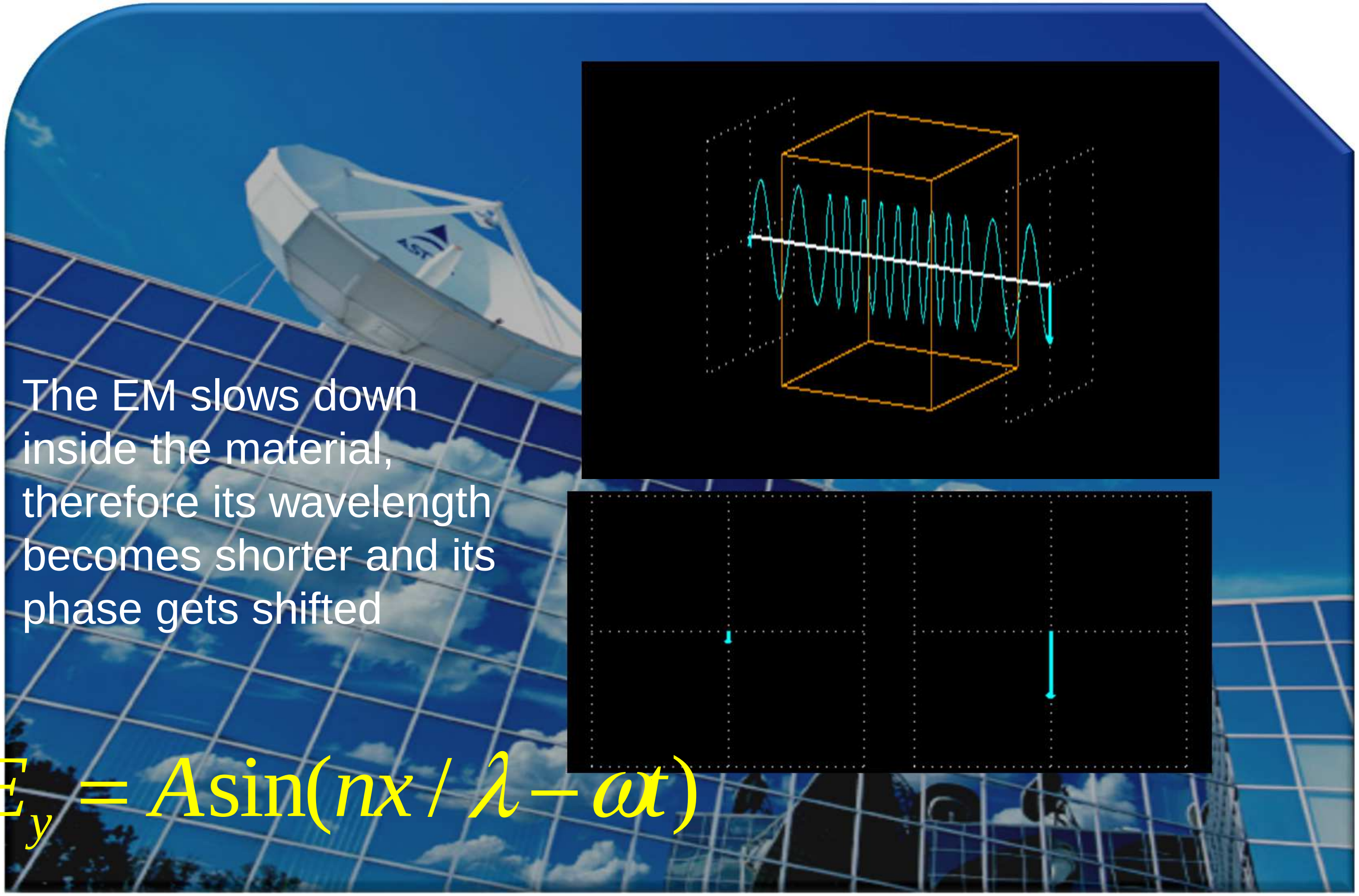


In

Out

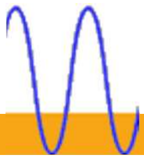
Refraction



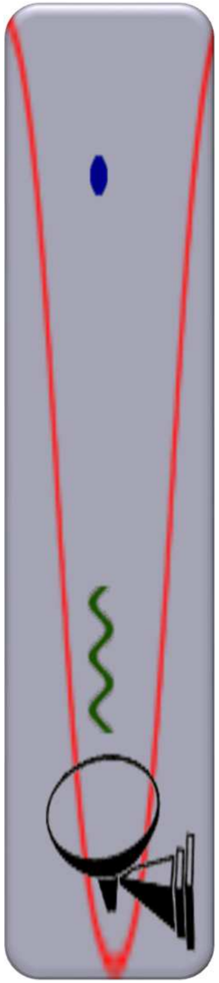


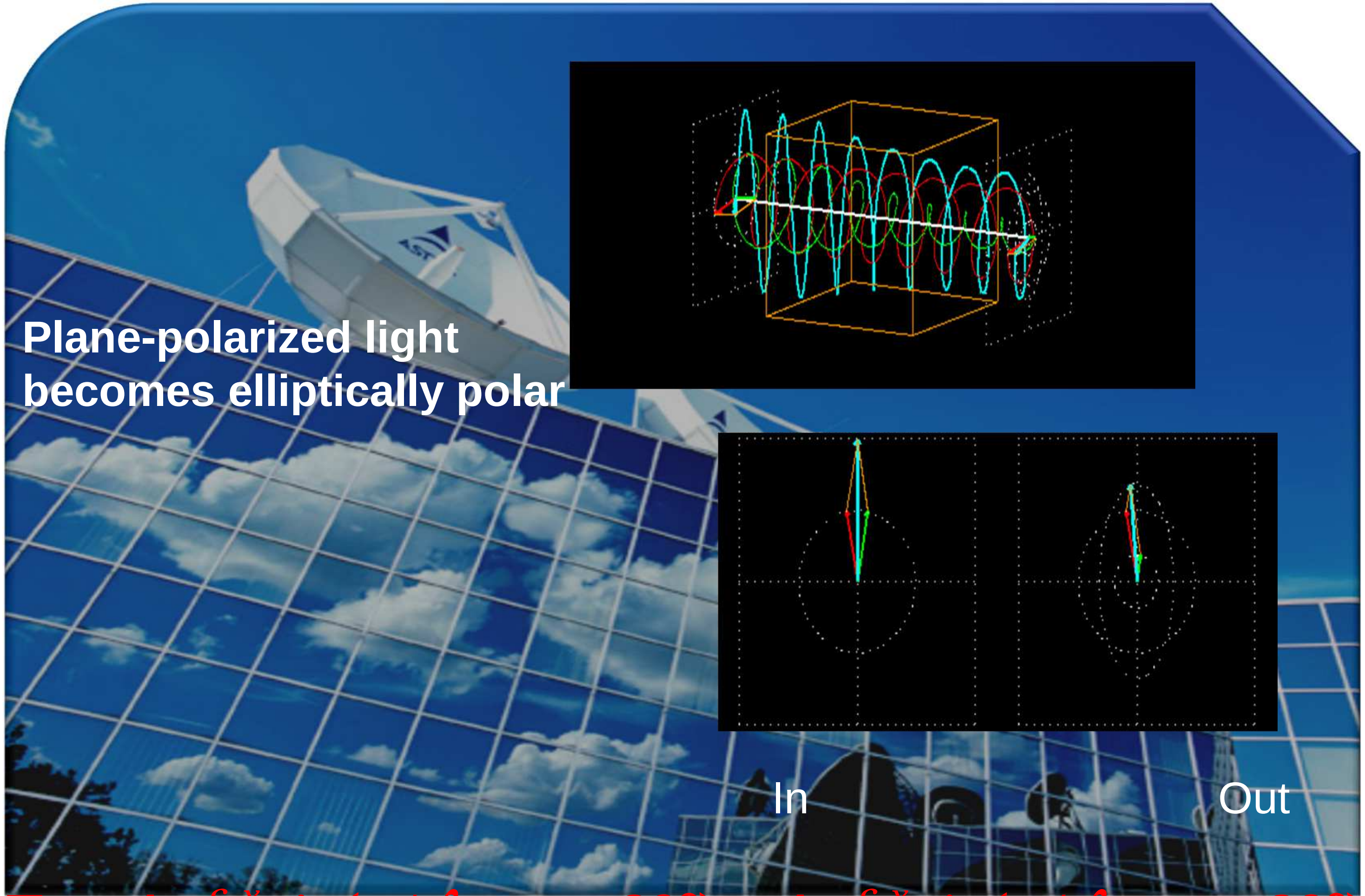
The EM slows down inside the material, therefore its wavelength becomes shorter and its phase gets shifted

$$E_y = A\sin(nx/\lambda - \omega t)$$



Circular dichroism





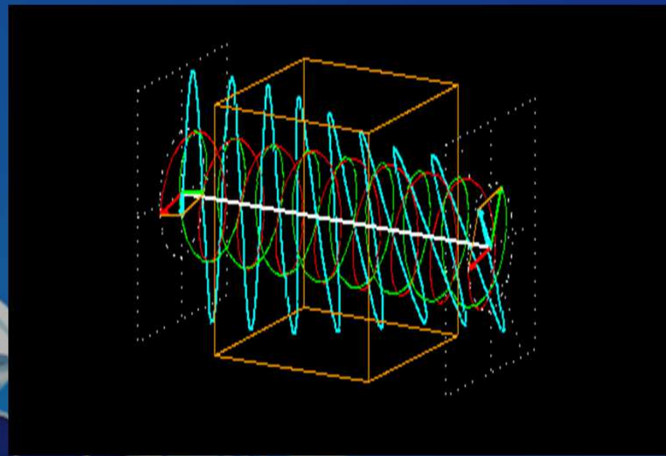
$$E_y = Ae^{-\varepsilon_R x}\sin(x/\lambda - \omega t + 90^\circ) + Ae^{-\varepsilon_L x}\sin(x/\lambda - \omega t - 90^\circ)$$

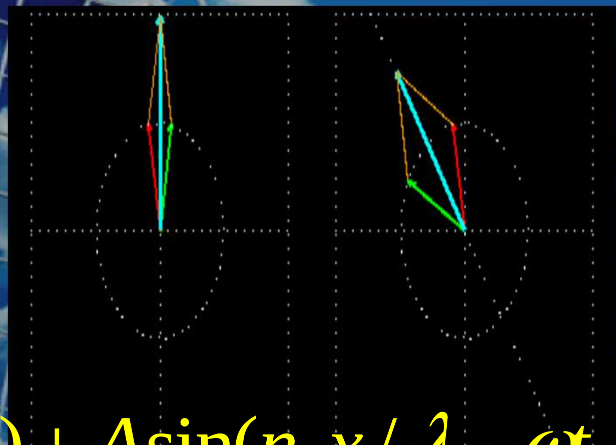
$$E_z = Ae^{-\varepsilon_R x}\sin(x/\lambda - \omega t) + Ae^{-\varepsilon_L x}\sin(x/\lambda - \omega t)$$

Circular bi-refringence

The plane of polarization of plane-polarized EM gets rotated

Material having different refraction indices for right and left circularly polarized EM: n_R and n_L

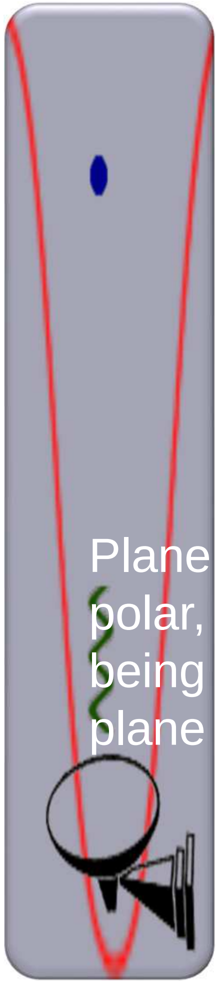




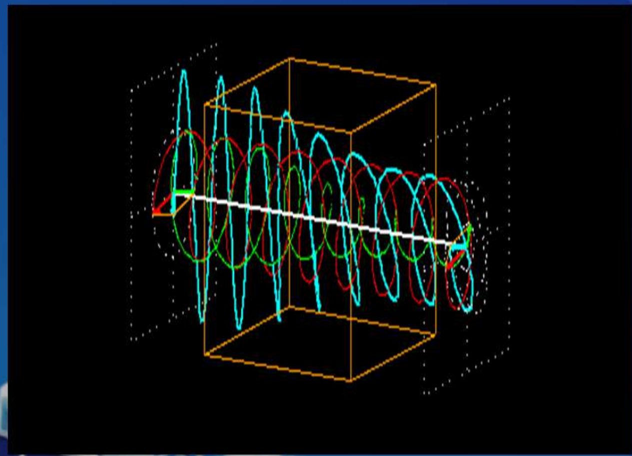
$$E_y = A\sin(n_R x/\lambda - \omega t + 90°) + A\sin(n_L x/\lambda - \omega t - 90°)$$

$$E_z = A\sin(n_R x/\lambda - \omega t) + A\sin(n_L x/\lambda - \omega t)$$

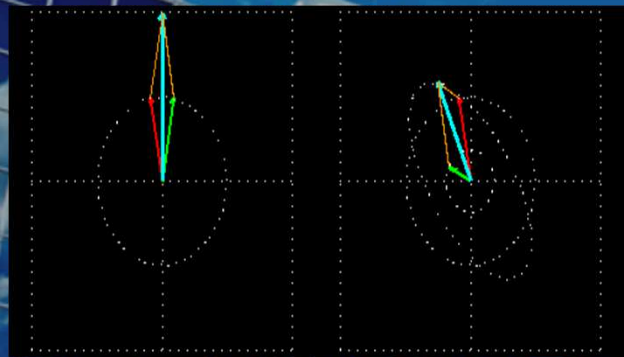
Circular dichroism AND bi-refringence



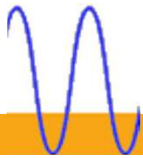
Material having different extincion coefficients AND refraction indices for right and left circularly polarized EM: ε_R and ε_L AND n_R and n_L



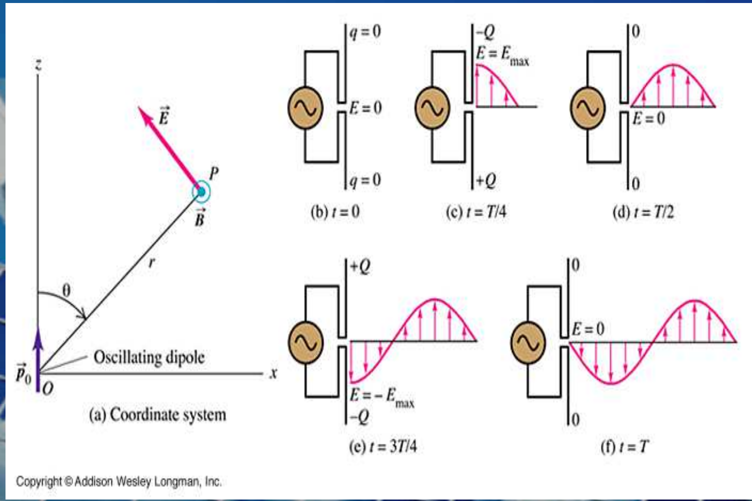
Plane larized EM gets elliptically polar, h the great axis of the ellipse being ated relative to the original plane polarization



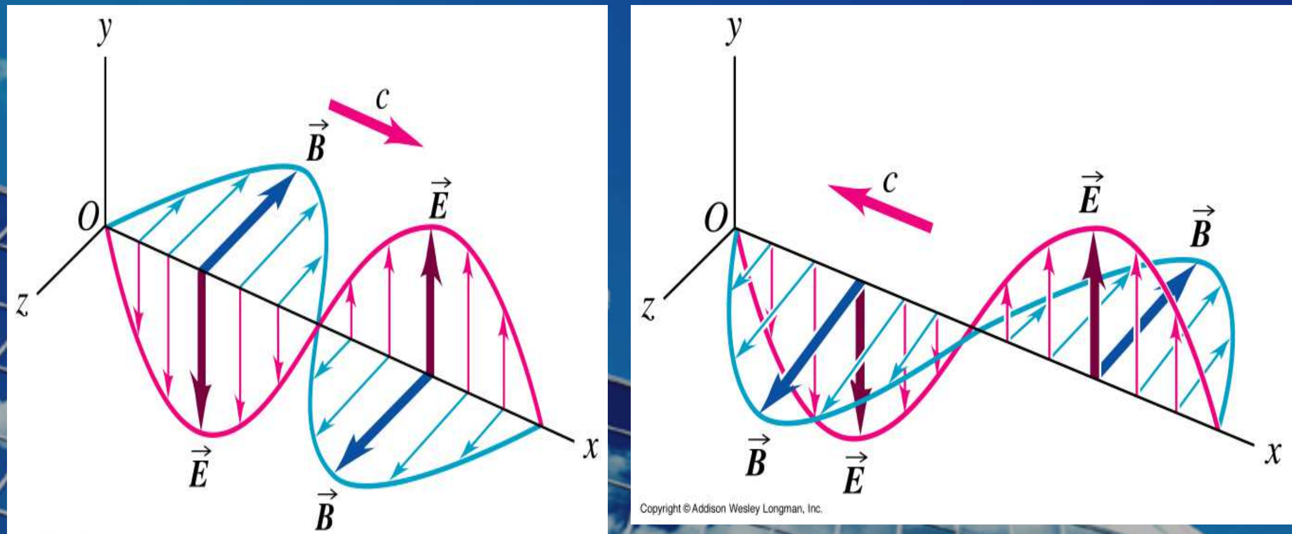
$$E_y = Ae^{-\varepsilon_R x}\sin(n_R x/\lambda - \omega t + 90°) + Ae^{-\varepsilon_L x}\sin(n_L x/\lambda - \omega t - 90°)$$



$$E_z = Ae^{-\varepsilon_R x}\sin(n_R x/\lambda - \omega t) + Ae^{-\varepsilon_L x}\sin(n_L x/\lambda - \omega t)$$



- One cycle in the production of an electro-magnetic wave by an oscillating electric dipole antenna.
- The red arrows represent the *E* field. (*B* not shown.)



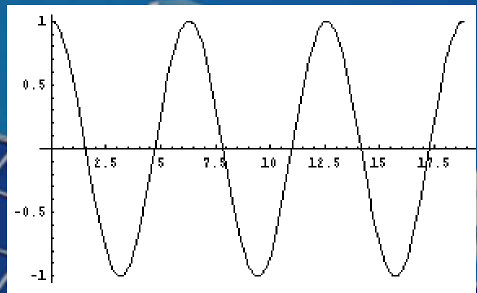
- Representation of the electric and magnetic fields in a propagating wave. One wavelength is shown at time *t* = 0.

- Propagation direction is ***E*** x ***B***.

Harmonic Plane Waves

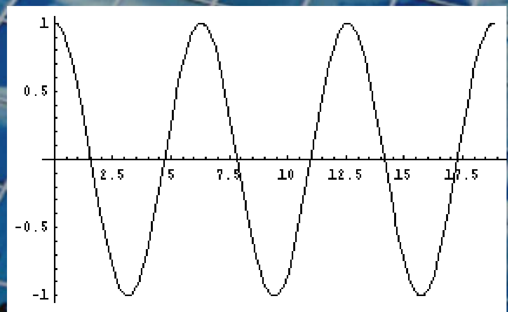
At t = 0

$\vec{E}$



At x = 0

$\vec{E}$



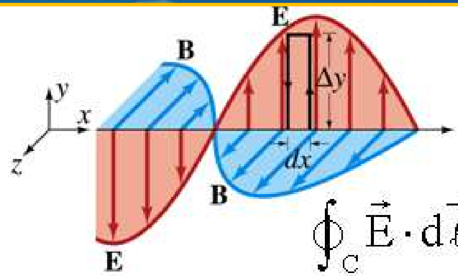
λ = spatial period or wavelength

phase velocity

$$v = \frac{\lambda}{T} = f\lambda = \frac{2\pi}{T}\frac{\lambda}{2\pi} = \frac{\omega}{k}$$

T = temporal period

Applying Faraday to radiation



$$\oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$$

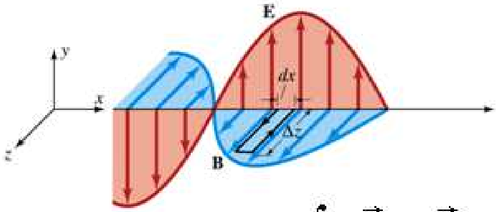
$$\oint_C \vec{E}\cdot d\vec{\ell} = \left(E + dE\right)\Delta y - E\Delta y = dE\Delta y$$

$$\frac{d\Phi_B}{dt} = \frac{dB}{dt}\,dx\Delta y$$

$$dE\Delta y = -\frac{dB}{dt}\,dx\Delta y$$

$$\frac{dE}{dx} = -\frac{dB}{dt}$$

Applying Ampere to radiation



$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_o \varepsilon_o \frac{d\Phi_E}{dt}$$

$$\oint_C \vec{B}\cdot d\vec{\ell} = B\Delta z - \left(B + dB\right)\Delta z = -dB\Delta z$$

$$\frac{d\Phi_E}{dt} = \frac{dE}{dt} dx\Delta z$$

$$-dB\Delta z = \mu_o \varepsilon_o \frac{dE}{dt} dx\Delta z$$

$$\frac{dB}{dx} = -\mu_o \varepsilon_o \frac{dE}{dt}$$

Fields are functions of both position (x) and time (t)

$$\frac{dE}{dx} = -\frac{dB}{dt}$$

$$\frac{dB}{dx} = -\mu_\circ \varepsilon_\circ \frac{dE}{dt}$$

Partial derivatives are appropriate

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

$$\frac{\partial B}{\partial x} = -\mu_\circ \varepsilon_\circ \frac{\partial E}{\partial t}$$

$$\frac{\partial^2 E}{\partial x^2} = -\frac{\partial}{\partial x}\frac{\partial B}{\partial t}$$

$$\frac{\partial}{\partial t}\frac{\partial B}{\partial x} = -\mu_\circ \varepsilon_\circ \frac{\partial^2 E}{\partial t^2}$$

$$\frac{\partial^2 E}{\partial x^2} = \mu_\circ \varepsilon_\circ \frac{\partial^2 E}{\partial t^2}$$

This is a wave equation!

The Trial Solution

- The simplest solution to the partial differential equations is a sinusoidal wave:
  - $E = E_{max} \cos(kx - \omega t)$
  - $B = B_{max} \cos(kx - \omega t)$
- The angular wave number is $k = 2\pi/\lambda$
  - λ is the wavelength
- The angular frequency is $\omega = 2\pi f$
  - f is the wave frequency

The Trial Solution

$$E = E_y = E_o \sin(kx - \omega t)$$

$$\frac{\partial^2 E}{\partial x^2} = \mu_o \varepsilon_o \frac{\partial^2 E}{\partial t^2}$$

$$\frac{\partial^2 E}{\partial x^2} = -k^2 E_o \sin(kx - \omega t) \qquad \frac{\partial^2 E}{\partial t^2} = -\omega^2 E_o \sin(kx - \omega t)$$

$$-k^2 E_o \sin(kx - \omega t) = -\mu_o \varepsilon_o \omega^2 E_o \sin(kx - \omega t)$$

$$\frac{\omega^2}{k^2} = \frac{1}{\mu_o \varepsilon_o}$$

The speed of light

$$v = \frac{\lambda}{T} = f\lambda = \frac{2\pi}{T}\frac{\lambda}{2\pi} = \frac{\omega}{k}$$

$$v = c = \frac{\omega}{k} = \sqrt{\frac{1}{\mu_o \varepsilon_o}}$$

Another look