EQUILIBRIUM OF RIGID BODIES

THREE DIMENSIONAL

Equilibrium of a particle

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$$

3 equations: Up to 3 unknowns

Equilibrium of a rigid body

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$$

$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0$$

6 equations: Up to 6 unknowns

Identify all relevant forces by drawing the Free–Body Diagram (FBD) of the rigid body.

FREE BODY DIAGRAM

A free body diagram, also called a force diagram is a pictorial representation often used by physicists and engineers to analyze the forces acting on a body of interest. A free body diagram shows all forces of all types acting on this body. Drawing such a diagram can aid in solving for the unknown forces or the equations of motion of the body. Creating a free body diagram can make it easier to understand the forces, and torques or moments, in relation to one another and suggest the proper concepts to apply in order to find the solution to a problem.

Procedure for drawing the FBD

- Draw the boundary of the chosen subset and detach/ separate it from all other bodies,
- For body with a mass, put the weight $= mg$ acting at the centre of gravity (G) of the body in the vertically downwards direction,
- Input all external forces acting on the body,

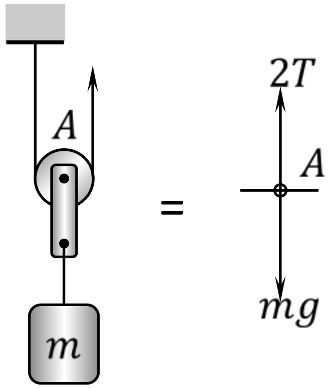
  For systems involving cables, the force of the cable *must* act outwards of the body, i.e. the cable must be in tension.
- place the reaction/s where the body touches or connected to a different rigid body,
- The FBD should also include dimensions for the process of taking moment

External Forces

Special Notes – Cables/ wires/ ropes and pulleys

- All cables are assumed to be inextensible.
- All forces acting from cables must direct outwards from point of analysis, i.e. in tension.
- When a cable passes a pulley, the tension is the same as long as it is the same cable.
- Pulleys are assumed to be smooth except stated otherwise.
- Dimensions of a pulley are usually neglected in calculations except stated otherwise.

Examples


A
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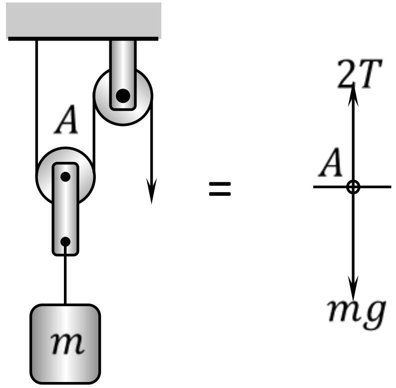
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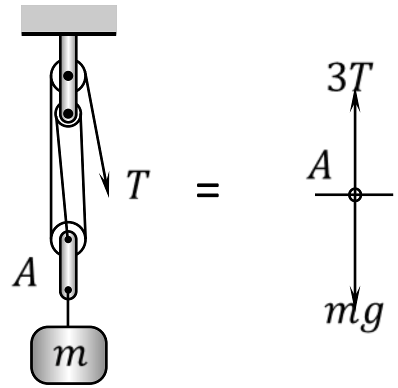
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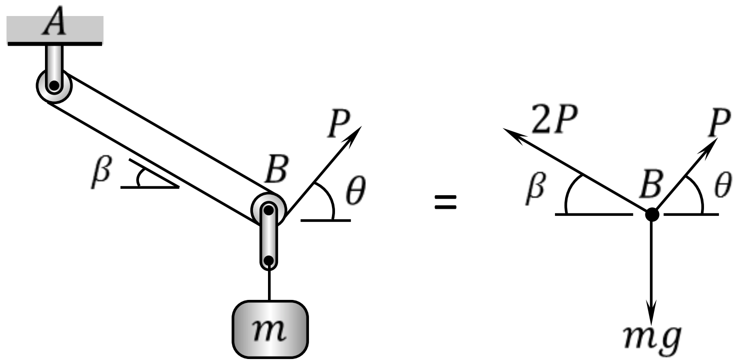
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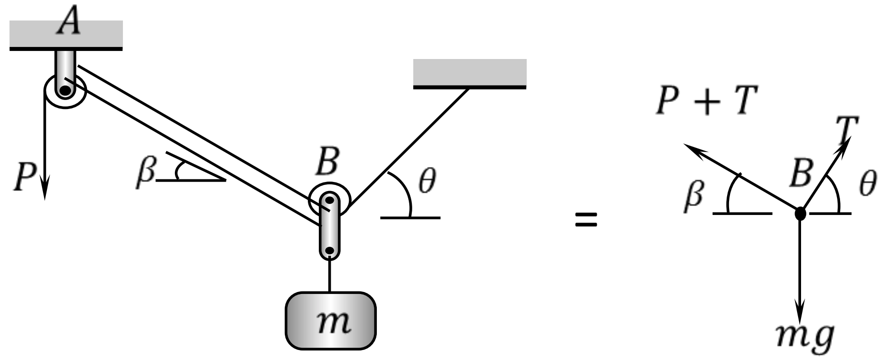
A
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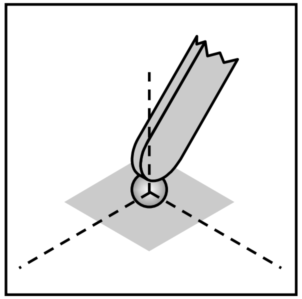
Reactions at Supports and Connections (3D)

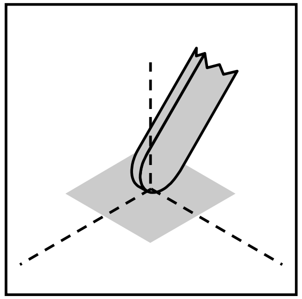
A force with a known line of action involving 1 unknown

Balls, Frictionless surface.

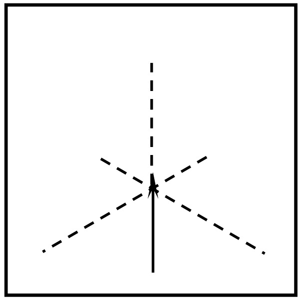
Perpendicular to the surface and must point towards the free body.



Ball



Frictionless surface



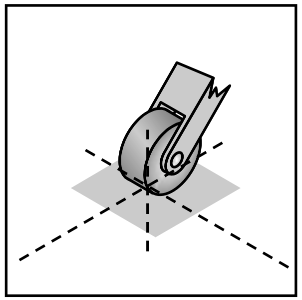
known line of action 1 unknown

Reactions at Supports and Connections (3D)

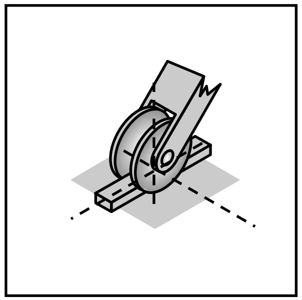
2 force components involving 2 unknowns

Roller on rough surface, Wheel on rail.

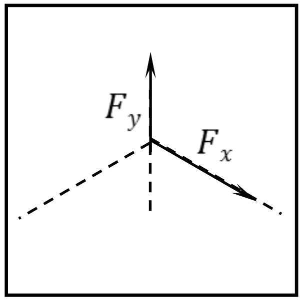
One perpendicular to the surface and must point towards the free body and the other 90° to it (tangent to the surface) that can be directed either way (not both), preventing translation in two directions.



rollers on rough surface



wheel on rail



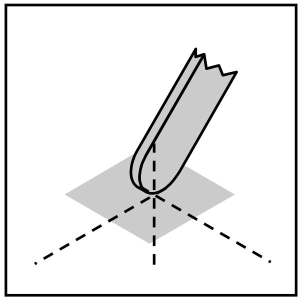
2 force components

Reactions at Supports and Connections (3D)

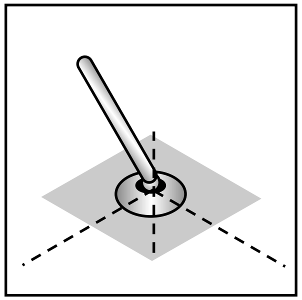
3 force components involving 3 unknowns

Rough surface, Ball and socket.

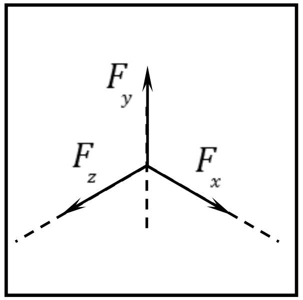
The force components are usually represented by their x, y and z components, preventing translation in three directions.



rough surface



ball and socket



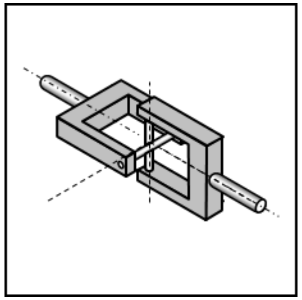
3 force components

Reactions at Supports and Connections (3D)

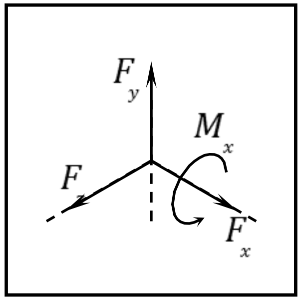
3 force components and 1 couple involving 4 unknowns

Universal Joint

The force components prevent translation in three directions. The universal joint is designed to allow rotation about two axes.



universal joint



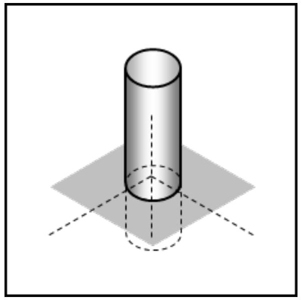
3 force components and 1 couple

Reactions at Supports and Connections (3D)

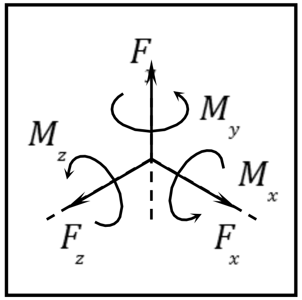
3 force components and 3 couples involving 6 unknowns

Fixed Support, Welded

The force components prevent translation in three directions. The couple components prevent rotation about three axes.



fixed support



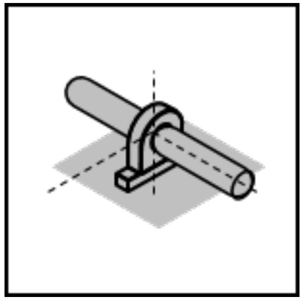
3 force components and 3 couples

Reactions at Supports and Connections (3D)

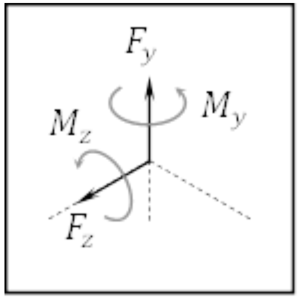
2 force components and (2 couples) involving 2 or (4) unknowns

Hinge and bearing supporting radial load only

Mainly designed to prevent translation in two directions, that may also include two couples. Generally will not exert any appreciable couple unless otherwise stated.



radial bearing



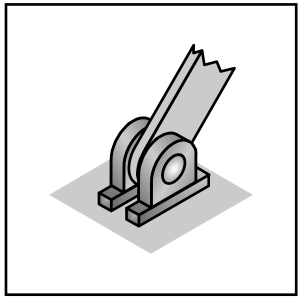
2 force components and (2 couples)

Reactions at Supports and Connections (3D)

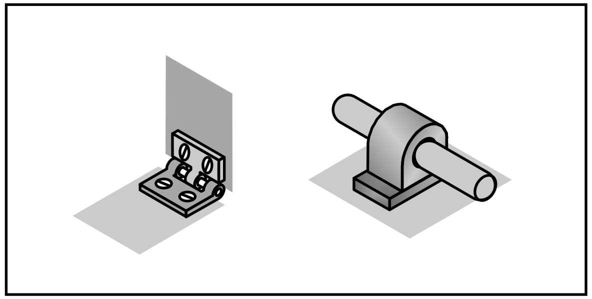
3 force components and (2 couples) involving 3 or (5) unknowns

Pin and bracket, Hinge and bearing supporting axial thrust and radial load

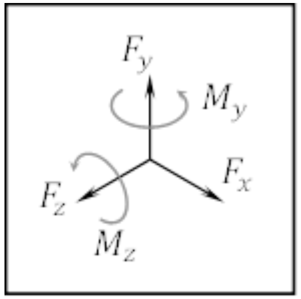
Designed to prevent translation in two directions, which may also include two couples. Generally will not exert any appreciable couple unless stated otherwise.



pin and bracket



hinge and bearing supporting axial thrust and radial load



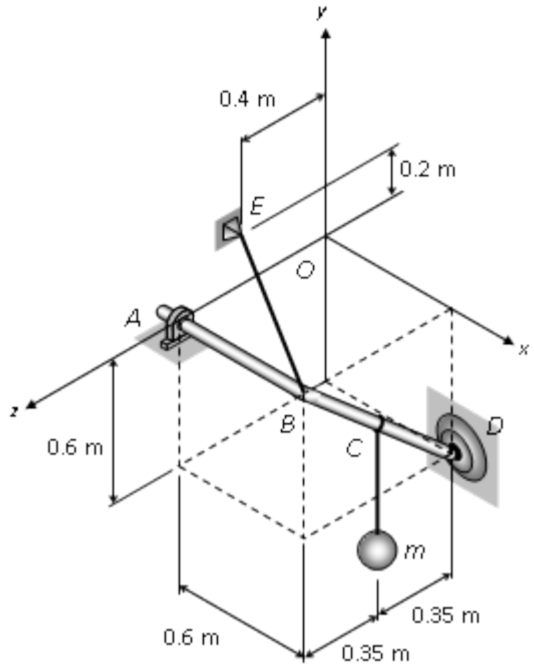
3 force components and (2 couples)

Important Notes

- Sketch the FBD for *every* solution.
- Knowledge on determination of force components, reactions and 'cross and dot products' (for taking moments) are essential.
- Since there are 6 equations, there can be up to 6 unknowns. For problems with more unknowns, try to solve using moment about an axis.

EXAMPLE 1

Determine the reaction at radial bearing A, the ball and socket joint D and the tension in cable BE to maintain mass m = 100 kg at the position shown. The bearing at A does not support any moment or axial load.



Addressing Forces

$$W = -100g\,\boldsymbol{j}$$

$$D = D_x\,\boldsymbol{i} + D_y\,\boldsymbol{j} + D_z\,\boldsymbol{k}$$

$$A = A_y\,\boldsymbol{j} + A_z\,\boldsymbol{k}$$

T_{BE}

$$\left.\begin{aligned} d_x &= -0.6 \\ d_y &= 0.2 \\ d_z &= -0.3 \end{aligned}\right\} \quad d = \sqrt{(0.6)^2 + (0.2)^2 + (-0.3)^2} = 0.7$$

$$T_{BE} = \frac{-0.6}{0.7} T_{BE}\,\boldsymbol{i} + \frac{0.2}{0.7} T_{BE}\,\boldsymbol{j} + \frac{-0.3}{0.7} T_{BE}\,\boldsymbol{k}$$

$$T_{BE} = -0.857 T_{BE}\,\boldsymbol{i} + 0.286 T_{BE}\,\boldsymbol{j} - 0.429 T_{BE}\,\boldsymbol{k}$$

$\sum F = 0$: Force Components

$\boldsymbol{i}$: $\quad D_x - 0.857 T_{BE} = 0 \quad$ (1)

$\boldsymbol{j}$: $\quad -100g + D_y + A_y + 0.286 T_{BE} = 0 \quad$ (2)

$\boldsymbol{k}$: $\quad D_z + A_z - 0.429 T_{BE} = 0 \quad$ (3)

Addressing moment

$$M_D = (r_{DC} \times -100g\,\boldsymbol{j}) + (r_{DB} \times T_{BE}) + (r_{DA} \times A) = 0$$

$$r_{DC} \times -100g\,\boldsymbol{j} = (0.3\,\boldsymbol{j} + 0.35\,\boldsymbol{k}) \times (-100g\,\boldsymbol{j}) = 343.35\ \text{N}\cdot\text{m}\,\boldsymbol{i}$$

$$\begin{aligned} r_{DB} \times T_{BE} &= (0.6\,\boldsymbol{j} + 0.7\,\boldsymbol{k}) \times (-0.857T_{BE}\,\boldsymbol{i} + 0.286T_{BE}\,\boldsymbol{j} - 0.429T_{BE}\,\boldsymbol{k}) \\ &= (0.6)(-0.857T_{BE})(-\boldsymbol{k}) + (0.6)(-0.429T_{BE})(\boldsymbol{i}) \\ &\quad + (0.7)(-0.857T_{BE})(\boldsymbol{j}) + (0.7)(0.286T_{BE})(-\boldsymbol{i}) \\ &= 0.514T_{BE}\,\boldsymbol{k} - 0.257T_{BE}\,\boldsymbol{i} - 0.6T_{BE}\,\boldsymbol{j} - 0.2T_{BE}\,\boldsymbol{i} \\ &= -0.457T_{BE}\,\boldsymbol{i} - 0.6T_{BE}\,\boldsymbol{j} + 0.514T_{BE}\,\boldsymbol{k} \end{aligned}$$

$$\begin{aligned} r_{DA} \times A &= (-0.6\,\boldsymbol{i} + 0.6\,\boldsymbol{j} + 0.7\,\boldsymbol{k}) \times \left(A_y\,\boldsymbol{j} + A_z\,\boldsymbol{k}\right) \\ &= (0.6)\left(A_y\right)(\boldsymbol{k}) + (-0.6)(A_z)(-\boldsymbol{j}) + (0.6)(A_z)(\boldsymbol{i}) + (0.7)\left(A_y\right)(-\boldsymbol{i}) \\ &= \left(0.6A_z - 0.7A_y\right)\boldsymbol{i} + 0.6A_z\,\boldsymbol{j} - 0.6A_y\,\boldsymbol{k} \end{aligned}$$

$\sum M = 0$: Moment Components

$\boldsymbol{i}$:	$343.35 - 0.457T_{BE} + (0.6A_z - 0.7A_y) = 0$		(4)
$\boldsymbol{j}$:	$-0.6T_{BE} + 0.6A_z = 0$	$A_z = T_{BE}$	(5)
$\boldsymbol{k}$:	$0.514T_{BE} - 0.6A_y = 0$	$A_y = 0.857T_{BE}$	(6)

input (5) and (6) into (4)

$$343.35 - 0.457T_{BE} + (0.6T_{BE} - 0.7(0.857T_{BE})) = 0$$

$$T_{BE} = 751.3 \text{ N}$$

$$A_z = 751.3 \text{ N}$$

$$A_y = 643.9 \text{ N}$$

substitute for T_{BE} into (1) gives

$D_x + -0.857\ (751.3) = 0$

$D_x = 643.9$ N

substitute for A_z and T_{BE} into (3)

$D_z + 751.3 - 0.429\ (751.3) = 0$

$D_z = -429$ N

substitute for A_y and T_{BE} into (2)

$$D_y + 643.9 + 0.286\ (751.3) - 100\ g = 0$$

$$D_y = 122.2\ \text{N}$$

Answers

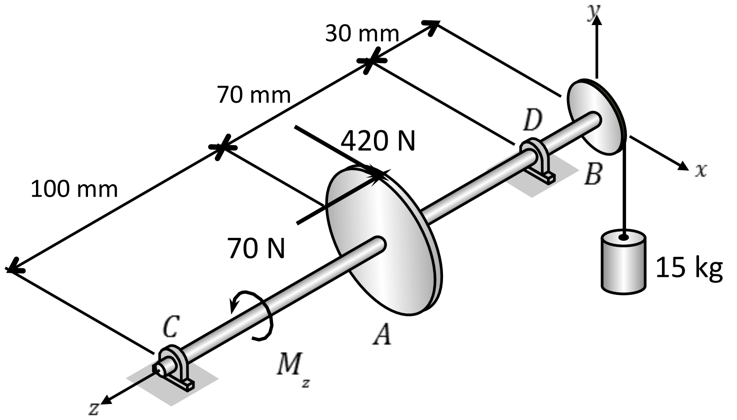
$$T_{BE} = 751.3\ \text{N}$$

$$A = 643.9\ \text{N}\ \boldsymbol{j} + 751.3\ \text{N}\ \boldsymbol{k}$$

$$D = 643.9\ \text{N}\ \boldsymbol{i} + 122.2\ \text{N}\ \boldsymbol{j} - 429\ \text{N}\ \boldsymbol{k}$$

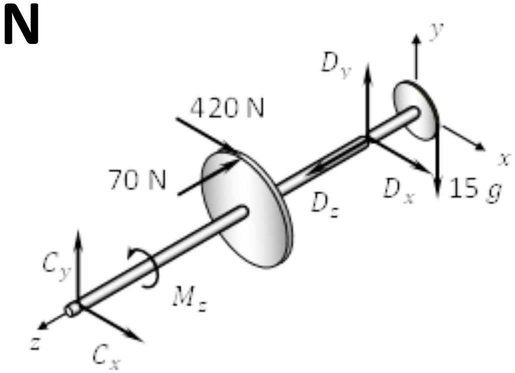
EXAMPLE 2

Gear A of 30 mm radius and pulley B of 15 mm radius are attached to a shaft supported by bearings C and D as shown. Both bearings at C and D do not support any moment and only bearing D supports axial load. Determine the required applied couple M_z and the reaction at bearings C and D for the system to maintain equilibrium.



SOLUTION

FBD



Addressing Forces

$C = C_x\,\boldsymbol{i} + C_y\,\boldsymbol{j}$

70 = – 70 $\boldsymbol{k}$

420 = 420 $\boldsymbol{i}$

$D = D_x\,\boldsymbol{i} + D_y\,\boldsymbol{j} + D_z\,\boldsymbol{k}$

15g = – 15g $\boldsymbol{j}$

$\sum F = 0$: Force Components

$\boldsymbol{i}$: $\quad C_x + 420 + D_x = 0 \qquad (1)$

$\boldsymbol{j}$: $\quad C_y + D_y - 15g = 0 \qquad (2)$

$\boldsymbol{k}$: $\quad -70 + D_z = 0 \qquad D_z = 70\text{ N}$

Addressing moment

$$M_D = M_z + (r_{DC} \times C) + (r_{DA} \times -70\,\boldsymbol{k}) + (r_{DA} \times 420\,\boldsymbol{i}) + (r_{DB} \times -15g\,\boldsymbol{j}) = 0$$

$$(r_{DC} \times C) = 0.17\,\boldsymbol{k} \times \left(C_x\,\boldsymbol{i} + C_y\,\boldsymbol{j}\right)$$
$$= -0.17C_y\,\boldsymbol{i} + 0.17C_x\,\boldsymbol{j}$$

$$(r_{DA} \times -70\,\boldsymbol{k}) = (0.03\,\boldsymbol{j} + 0.07\,\boldsymbol{k}) \times (-70\,\boldsymbol{k})$$
$$= (0.03)(-70)(\boldsymbol{i}) = -2.1\,\boldsymbol{i}$$

$$(r_{DA} \times 420\,\boldsymbol{i}) = (0.03\,\boldsymbol{j} + 0.07\,\boldsymbol{k}) \times (420\,\boldsymbol{i})$$
$$= (0.03)(420)(-\boldsymbol{k}) + (0.07)(420)(\boldsymbol{j})$$
$$= 29.4\,\boldsymbol{j} - 12.6\,\boldsymbol{k}$$

$$\left(r_{DB} \times -15g\,\boldsymbol{j}\right) = (0.015\,\boldsymbol{i} - 0.03\,\boldsymbol{k}) \times (-15g\,\boldsymbol{j})$$
$$= (0.015)(-15g)(\boldsymbol{k}) + (-0.03)(-15g)(-\boldsymbol{i})$$
$$= -4.4145\,\boldsymbol{i} - 2.20725\,\boldsymbol{k}$$

$\sum M = 0$: Moment Components

$\boldsymbol{i}$:	$-0.17C_y - 2.1 - 4.4145 = 0$	$C_y = -38.3$ N
$\boldsymbol{j}$:	$0.17C_x + 29.4 = 0$	$C_x = -172.9$ N
$\boldsymbol{k}$:	$M_z - 12.6 - 2.20725 = 0$	$M_z = 14.80725$ N

substitute for C_x into (1)

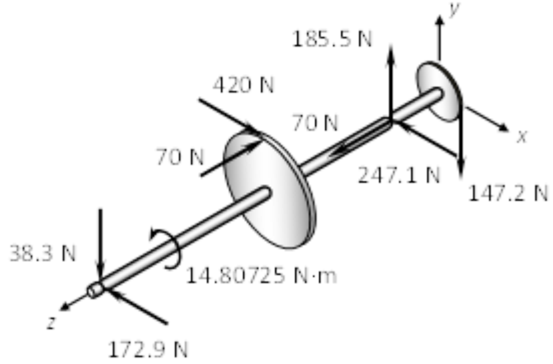
$$-172.9 + 420 + D_x = 0$$
$$D_x = -247.1 \text{ N}$$

substitute for C_y into (1)

$$-38.3 + D_y - 15g = 0$$
$$D_y = 185.5 \text{ N}$$

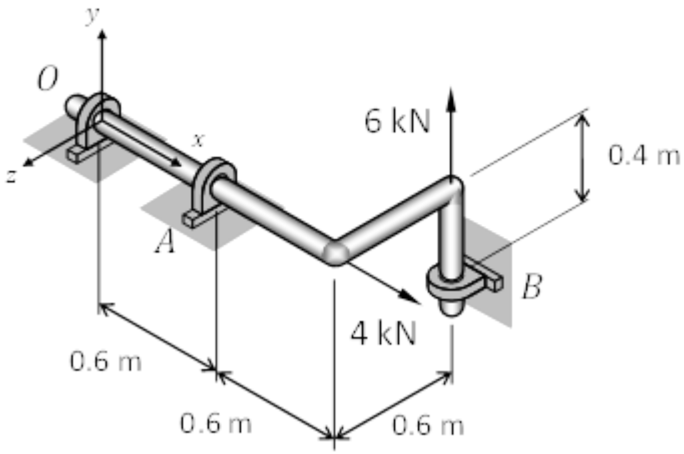
Answers

M_z = 14.80725 N·m
C = – 172.9 N $\boldsymbol{i}$ – 38.3 N $\boldsymbol{j}$
D = – 247.1 N $\boldsymbol{i}$ + 185.5 N $\boldsymbol{j}$ + 70 N $\boldsymbol{k}$



QUESTION 1

The bent rod shown is subjected to two forces and supported by radial bearings at O, A and B. The bearings do not support any axial load or moment. Determine components of the reactions at these supports.



QUESTION 2

The bent rod shown is supported by cable CE, a ball and socket joint at D, and a radial bearing at A. The bearing at A does not support any axial thrust or couple. Determine the tension in cable CE and the reaction at A and D when a load P = 10 kN is applied at B.
All dimensions in m.