SEB4233
Biomedical Signal Processing

Discrete-Time and System (A Review)

Dr. Malarvili Balakrishnan

Discrete – Time Signal

- A discrete-time signal, also referred as ***sequence***, is only defined at discrete time instances.

- A function of discrete time instants that is defined by integer n.

$$x(t) \longrightarrow x(n)$$

Continuous time signal ↑ $x(t)$ Discrete time signal ↑ $x(n)$

Periodic and Non-Periodic

- Periodic signal is

$$x(n) = x(n+N) \qquad -\infty < n < \infty$$

- Non-periodic (aperiodic) signal is

$x(n)$ defined within $-N/2 \le n \le N/2$, and $N < \infty$

Examples Of Aperiodic Signals

▪ Impulse function	$x(n)=1$	$n=0$
	$=0$	elsewhere
▪ Step function	$x(n)=1$	$n \geq 0$
	$=0$	$n < 0$
▪ Ramp Function	$x(n)=a$	$n \geq 0$
	$=0$	$n < 0$
▪ Pulse function	$x(n)=1$	$n_0 \leq n \leq n_1$
	$=0$	elsewhere
▪ Pulse sinusoid	$x(n)=\cos(2\pi n f_1 - \phi)$	$n_0 \leq n \leq n_1$
	$=0$	elsewhere

Energy and Power

- Energy

$$E_x = \sum_{n=0}^{N-1} x(n)x^*(n) = \sum_{n=0}^{N-1} \left|x(n)\right|^2$$

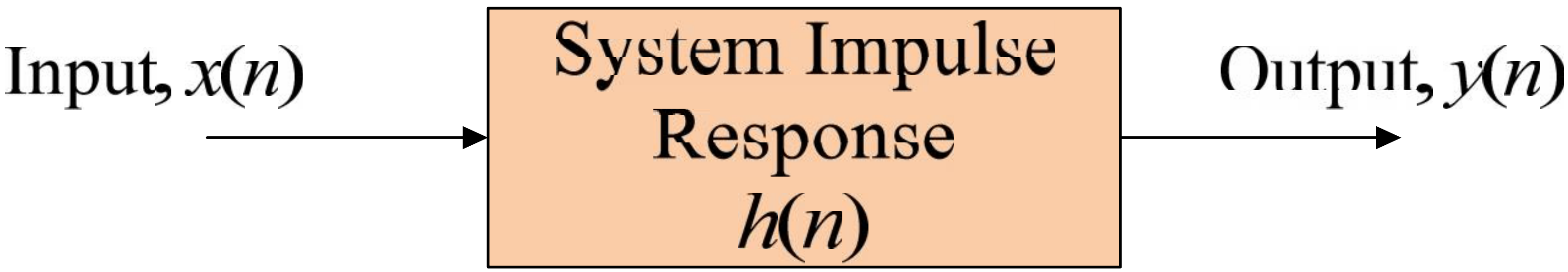
- Power

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} x(n)x^*(n) = \frac{1}{N}\sum_{n=0}^{N-1} \left|x(n)\right|^2 = \frac{1}{N} E_x$$

where N is the duration of the signal

Systems

- A system operates on a signal to produce an output.



Characteristics of systems

- time invariant
- shift invariant
- causal
- stability
- Linearity

- Time and shift invariant means that the system characteristics and shift do not change with time.

Characteristics of systems

- Causality

$$h(n) \neq 0 \qquad n \geq 0$$
$$= 0 \qquad n < 0$$

- Stability

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

- Linearity

$$T[x_0(n)] + T[x_1(n)] = T[x_0(n) + x_1(n)]$$ $x_0(n)$ & $x_1(n)$ are 2 different inputs

$$y(n) = S[T[x(n)]] = T[S[x(n)]]$$ $S[\ ]$ and $T[\ ]$ are linear transformations

Convolution

- If $h(n)$ is the system impulse response, then the input-output relationship is a convolution.
- It is used for designing filter or a system.
- Definition of convolution:

$$y(n) = h(n) * x(n) = \sum_{\lambda=-\infty}^{\infty} h(\lambda)x(n-\lambda)$$

$$y(n) = x(n) * h(n) = \sum_{\lambda=-\infty}^{\infty} x(\lambda)h(n-\lambda)$$

Example

Consider a system with an impulse response of

$$h(n) = [\ 1\ 1\ 1\ 1\ ]$$

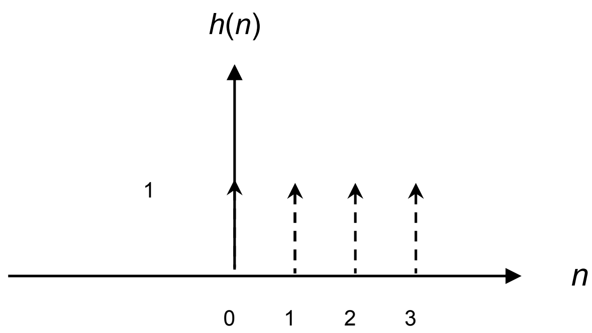
If the input to the signal is

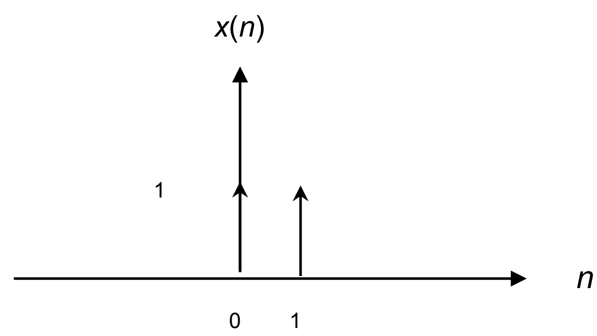
$$x(n) = [\ 1\ 1\ ]$$

- Thus, the output of the system is $$y(n) = \sum_{\lambda=-\infty}^{\infty} h(\lambda)x(n-\lambda)$$

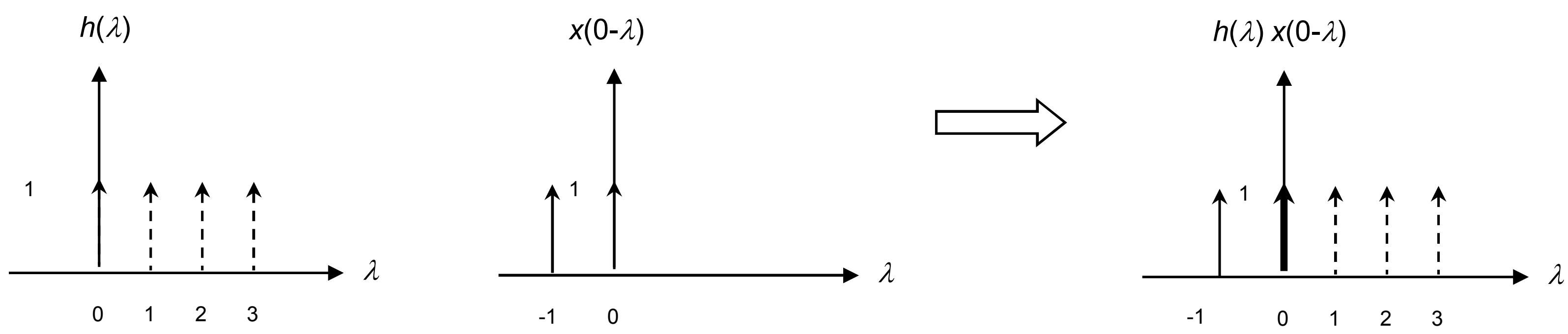
- The result of the convolution procedure in its graphical form is :

Example (Cont.)





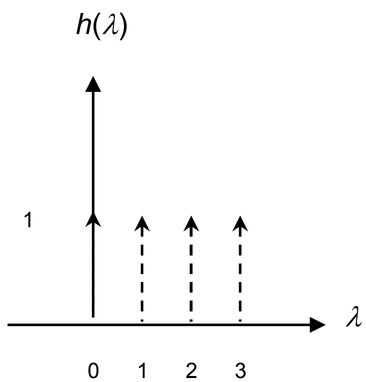
i) The definition of the system impulse response *h*(*n*) and the input signal *x*(*n*)

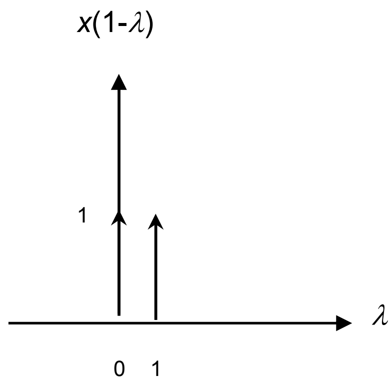


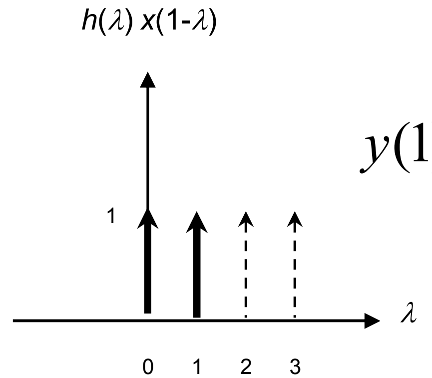
$$y(0) = \sum_{\lambda=-\infty}^{\infty} h(\lambda)x(0-\lambda) = 1$$

Example (Cont.)

ii) The result at n=1.

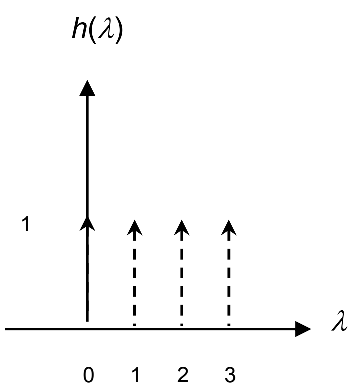


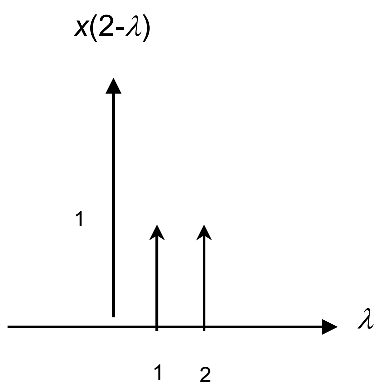


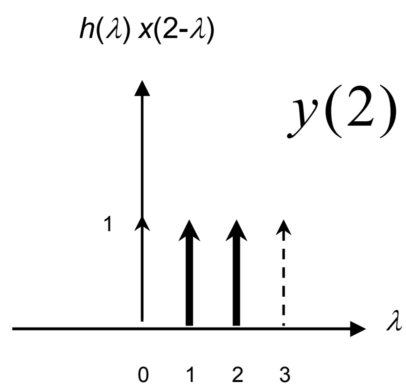


$$y(1) = \sum_{\lambda=-\infty}^{\infty} h(\lambda)x(1-\lambda) = 2$$

iii) The result at n=2.

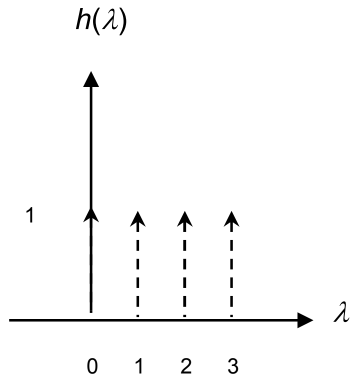


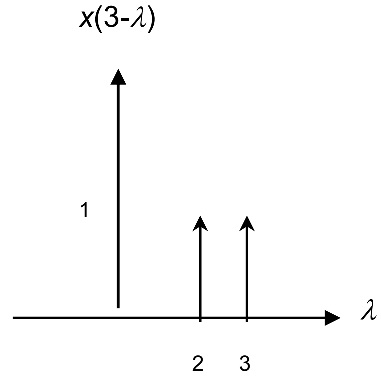


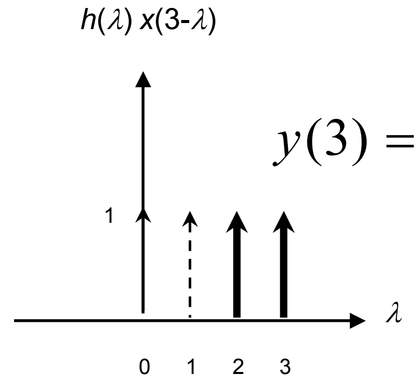


$$y(2) = \sum_{\lambda=-\infty}^{\infty} h(\lambda)x(2-\lambda) = 2$$

iV) At n=3

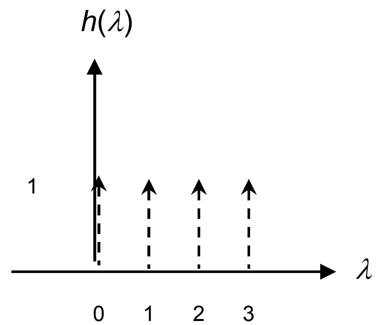


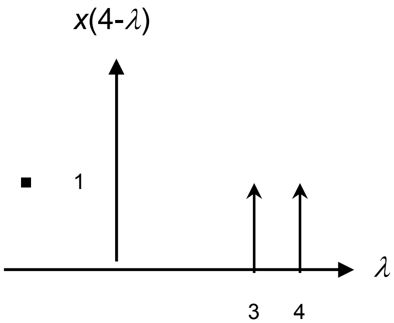


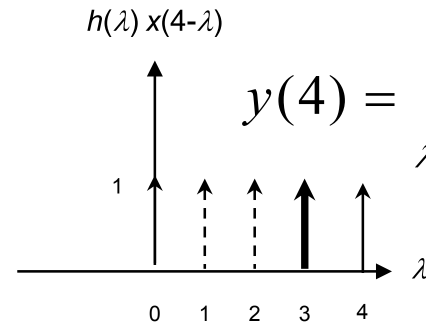


$$y(3) = \sum_{\lambda=-\infty}^{\infty} h(\lambda)x(3-\lambda) = 2$$

v) At n=4

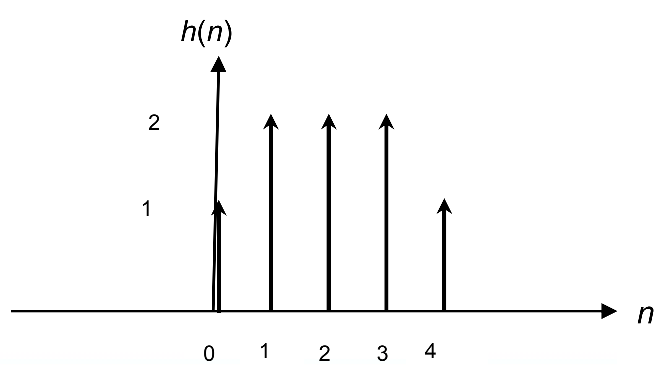






$$y(4) = \sum_{\lambda=-\infty}^{\infty} h(\lambda)x(4-\lambda) = 1$$

Finally



Frequency Domain Representation

- An alternative representation and characterization of signals.
- Much more information can be extracted from a signal.
- Many operations that are complicated in time domain become rather simple.

- Fourier Transforms:
  - Fourier series – for periodic continuous time signals
  - Fourier Transform – for aperiodic continuous time signals

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega)$$

  - Discrete Time Fourier Transform (DTFT) – for aperiodic discrete time signals (frequency domain is still continuous however)

$$x(n) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

  - Discrete Fourier Transform (DFT) – DTFT sampled in the frequency domain
  - Fast Fourier Transform (FFT) – Same as DFT, except calculated very efficiently

DTFT & its Inverse

- Since the sum of $x[n]$, weighted with continuous exponentials, is continuous, the DTFT $X(\omega)$ is continuous (non-discrete)
- Since $X(\omega)$ is continuous, $x[n]$ is obtained as a continuous integral of $X(\omega)$, weighed by the same complex exponentials.
- $x[n]$ is obtained as an integral of $X(\omega)$, where the integral is over an interval of **2pi.**
- $X(\omega)$ is sometimes denoted as $X(e^{j\omega})$ in some books.

$$x[n] \overset{\Im}{\Leftrightarrow} X(\omega)$$

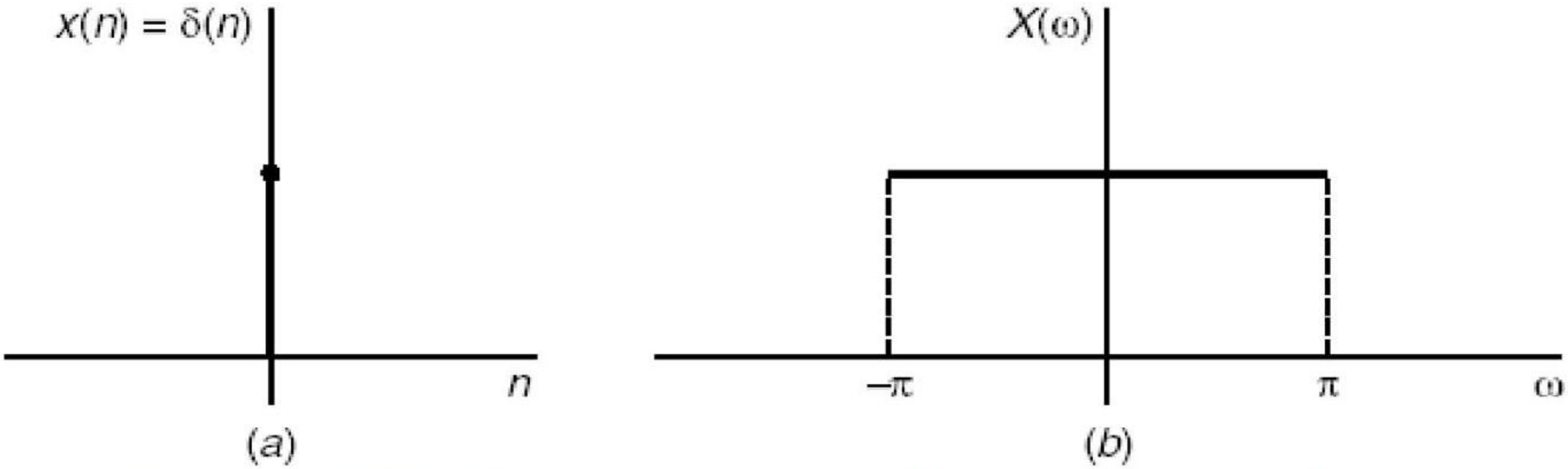
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$\overset{\Im}{\Leftrightarrow}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Example: DTFT of Impulse Function

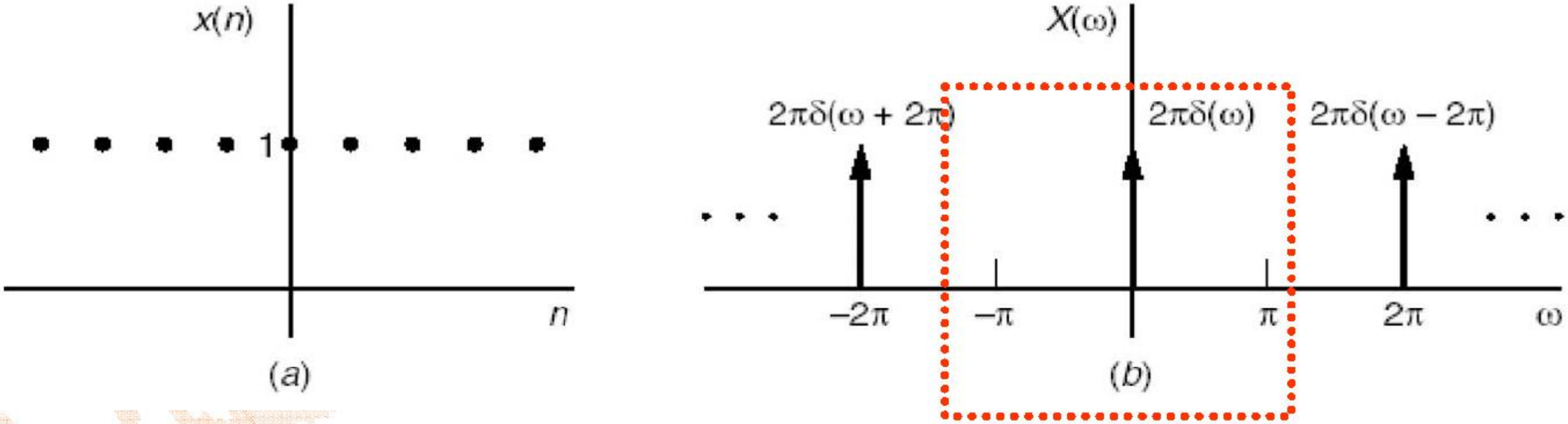
$$\Im\{\delta[n]\} = 1$$



Extend of the frequency band
in discrete frequency domain

Example: DTFT of constant function

$$\Im\{1\} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m)$$



Discrete Fourier Transform

- DTFT does not involve any sampling- it's a continuous function
- Not possible to determine DTFT using computer
- So explore another way to represent discrete-time signals in frequency domain
- The exploration lead to DFT

DFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{+\frac{j2\pi kn}{N}} \qquad 0 \le n \le N-1 \qquad x(n) \rightarrow Aperiodic\ , discrete\text{ - }time$$

$$=0 \qquad \text{elsewhere}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{\frac{-j2\pi kn}{N}} \qquad 0 \le k \le N-1 \qquad X(k) \rightarrow \text{Aperiodic, discrete} - \text{frequency}$$

$$=0 \qquad \text{elsewhere}$$

Energy and Power Spectrum

- Energy

$$E_{xx}(k) = \left| X(k) \right|^2$$

- Power Spectrum

$$S_{xx}(k) = \frac{1}{N} \left| X(k) \right|^2$$

Fast Fourier Transform

- The computation complexity of the *N* length DFT is *N*2.
- The FFT (Fast Fourier Transform) is developed to reduce the computation complexity to *N ln* (*N*).
- Now can implement frequency domain processing in real-time.

FFT

- The two approaches for implementing the FFT:

  - Decimation in Time (DIT):

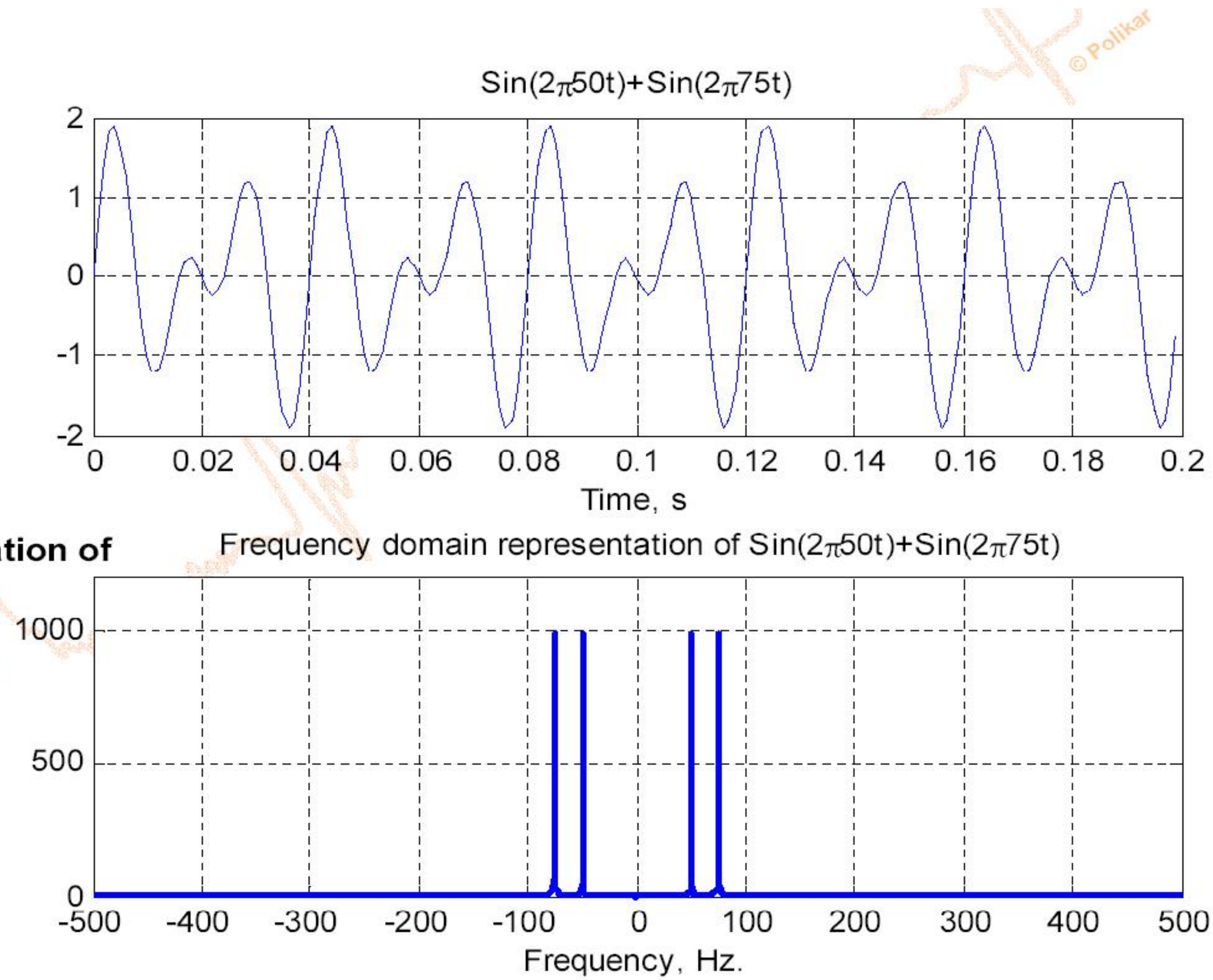
$$X(k) = \sum_{n=even} x(n)\exp\left(-j\frac{2\pi kn}{N}\right) + \sum_{n=odd} x(n)\exp\left(-j\frac{2\pi kn}{N}\right)$$

  - Decimation in Frequency (DIF):

$$X(k) = \sum_{n=0}^{N/2-1} x(n)\exp\left(-j\frac{2\pi kn}{N}\right) + \sum_{n=N/2}^{N-1} x(n)\exp\left(-j\frac{2\pi kn}{N}\right)$$

FFT in Matlab

```
t=-1:0.001:1;
x=sin(2*pi*50*t)+sin(2*pi*75*t);
subplot(211)
plot(t(1001:1200),x(1:200))
grid
title('Sin(2\pi50t)+Sin(2\pi75t)')
xlabel('Time, s')
subplot(212)
X=abs(fft(x));
X2=fftshift(X);
f=-499.9:1000/2001:500;
plot(f,X2);
grid
title(' Frequency domain representation of Sin(2\pi50t)+Sin(2\pi75t)')
xlabel('Frequency, Hz.')
```



Convolution in Frequency Domain

- Convolution in time domain = multiplication in frequency domain

$$x[n] * h[n] \overset{\Im}{\Leftrightarrow} X(\omega) \cdot H(\omega)$$

- Multiplication in time domain = convolution in frequency domain

$$x[n] \cdot h[n] \overset{\Im}{\Leftrightarrow} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\gamma) H(\omega - \gamma) d\gamma$$

Sampling

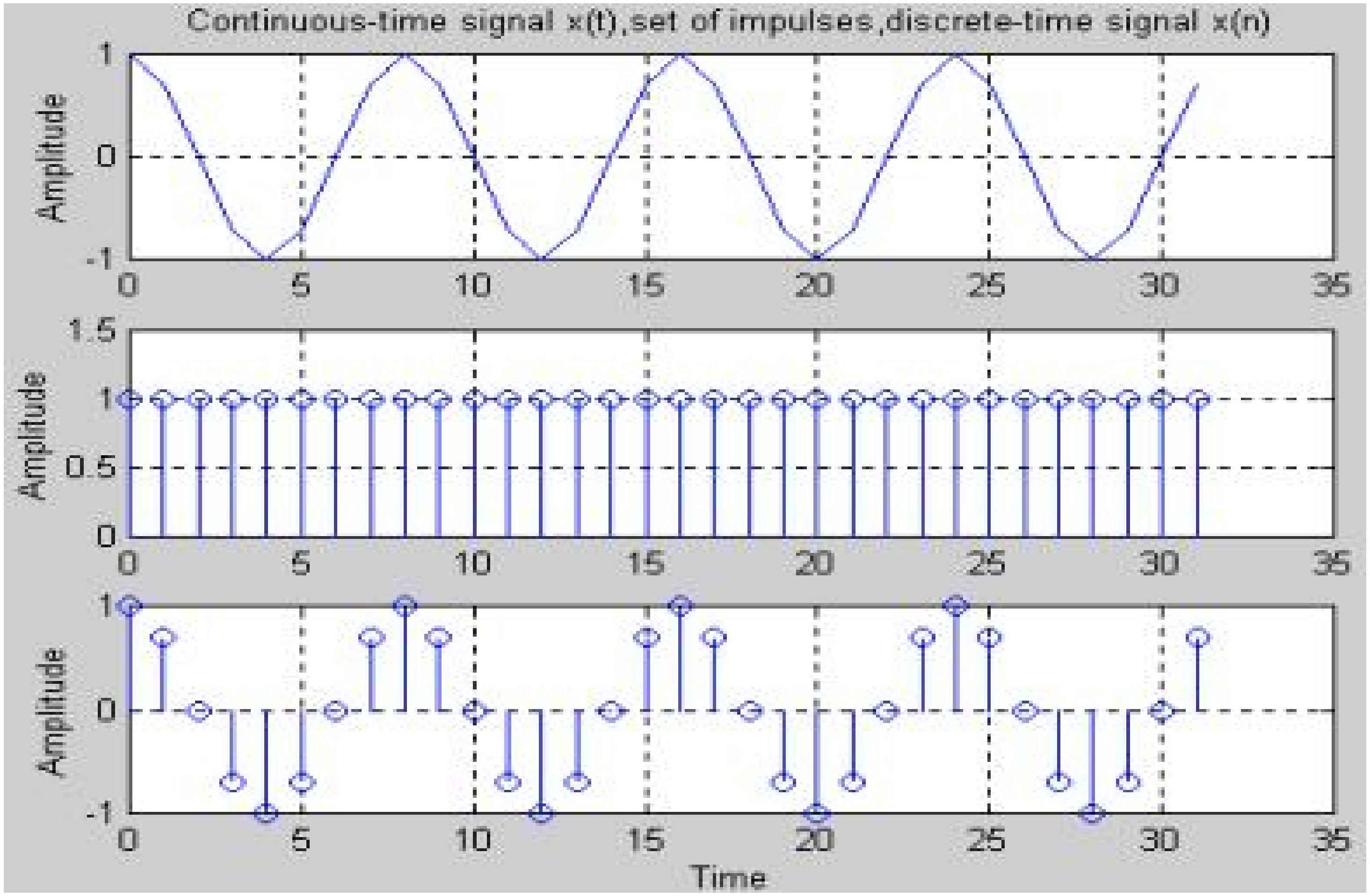
- Sampling: Process of conversion from continuous-time to discrete-time representation.

- This is necessary if it is desired to process the signal using digital computers.

- The discrete-time signal $x(n)$ is obtained as a result of the product of the continuous-time signal with a set of impulse $x_d(t)$ with period T_s

$$x(n) = x(t)x_{\delta}(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s)$$

Sampling



Sampling Process

Spectrum of Sampled Signals

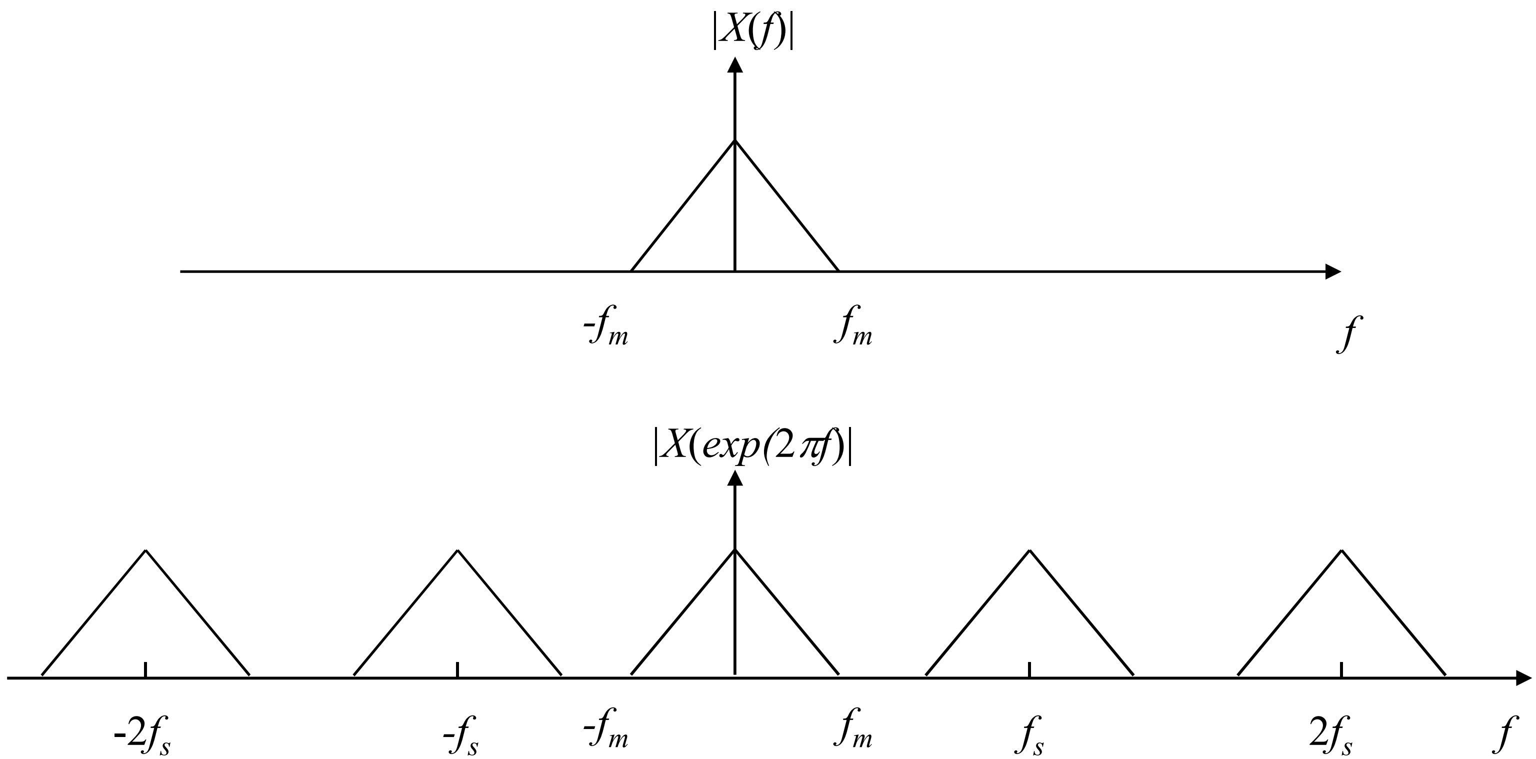
- If $x(t)$ has a spectrum $X(f)$, then the spectrum of a sampled signal $x(n)$ is

$$X(\exp(j2\pi f)) = FT[x(n)] = FT\left[x(t)x_\delta(t)\right] = FT\left[\sum_{n=-\infty}^{\infty} x(t)\delta(t-nT_s)\right]$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t)\delta(t-nT_s)\exp(-j2\pi ft)dt = \sum_{n=-\infty}^{\infty} x(n)\exp(-j2\pi fnT_s)$$

$$= \sum_{n=-\infty}^{\infty} X(f-nf_s)$$

Spectrum of Sampled Signals



Amplitude spectra of a signal before and after sampling.

Nyquist Sampling Theorem

- Increasing the sampling frequency will increase the storage space and processing time.
- Reducing the sampling frequency will result in aliasing due to the overlapping between the desired and replicate spectrum components.
- The aliasing effect is minimized by using the Nyquist sampling theorem

$$f_s \geq 2 f_{\max}$$

Difference Equations

- A continuous-time system can be described by differential equations.
- Discrete-time systems are described by difference equations that can be expressed in general form as

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + \cdots a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

Outputs y[n] Inputs x[n]

$$\sum_{i=0}^{N} a_i y[n-i] = \sum_{j=0}^{M} b_j x[n-j], \quad a_0 = 1$$

Constant coefficients

- Constant coefficients a_i and b_i are called filter coefficients.
- Integers M and N represent the maximum **delay** in the input and output, respectively. The larger of the two numbers is known as the **order of the filter**.

Difference Equations

$$y(n)+\sum_{\lambda=1}^{M}a(\lambda)y(n-\lambda)=\sum_{\lambda=0}^{M}b(\lambda)x(n-\lambda)$$ (Infinite Impulse Response - IIR)

$$y(n)=\sum_{\lambda=0}^{M}b(\lambda)x(n-\lambda)$$ (Finite Impulse Response - FIR)