SKMM 3033
Finite Element Method

Topic 2: Matrix algebra and Gaussian Elimination Method

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By the end of the notes:

The students are expected:

- To understand matrix manipulation to solve simultaneous equations
- To apply Gaussian elimination to solve simultaneous equations

Solving Simultaneous Equations

Example:

$$2x + y = 7 \qquad (1)$$

$$3x - y = 8 \qquad (2)$$

Solve for x and y.

!! Two options to solve the problem !!

Solving Simultaneous Equations

Example:

$$2x + y = 7 \qquad (1)$$

$$3x - y = 8 \qquad (2)$$

Solve for x and y.

Option 1: Substitution

Rearrange (1):

$$y = 7 - 2x \qquad (3)$$

Substitute (3) to (2)

$$3x - (7 - 2x) = 8$$

$$3x + 2x = 8 + 7$$

$$5x = 15$$

$$\therefore x = 3$$

Finally,

$$y = 7 - 2(3)$$

$$\therefore y = 1$$

Solving Simultaneous Equations

Example:

$$2x + y = 7 \qquad (1)$$

$$3x - y = 8 \qquad (2)$$

Solve for x and y.

Option 2: Inversion of coefficient matrix

$$\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

$$AX = B$$

Where

$$A = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}; \quad X = \begin{pmatrix} x \\ y \end{pmatrix}; \quad B = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

Solving Simultaneous Equations

Example:

$$2x + y = 7 \qquad (1)$$

$$3x - y = 8 \qquad (2)$$

Solve for x and y.

Option 2: Inversion of coefficient matrix

$$AX = B$$

$$\therefore X = A^{-1}B$$

!! We need to calculate inverse matrix A !!

Solving Simultaneous Equations

Example:

$$2x + y = 7 \qquad (1)$$

$$3x - y = 8 \qquad (2)$$

Solve for x and y.

Option 2: Inversion of coefficient matrix

Inverse Matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Determinant: $|A| = ad - bc$

Solving Simultaneous Equations

Example:

$$2x + y = 7 \qquad (1)$$

$$3x - y = 8 \qquad (2)$$

Solve for x and y.

Option 2: Inversion of coefficient matrix

Inverse Matrix:

$$A^{-1} = \frac{1}{2\times(-1) - 1\times 3}\begin{pmatrix} -1 & -1 \\ -3 & 2 \end{pmatrix}$$

$$= -\frac{1}{5}\begin{pmatrix} -1 & -1 \\ -3 & 2 \end{pmatrix}$$

Solving Simultaneous Equations

Example:

$$2x + y = 7 \qquad (1)$$

$$3x - y = 8 \qquad (2)$$

Solve for x and y.

Option 2: Inversion of coefficient matrix

$$X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -1 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

$$= -\frac{1}{5}\begin{pmatrix} (-1\times 7)+(-1\times 8) \\ (-3\times 7)+(2\times 8) \end{pmatrix}$$

$$= -\frac{1}{5}\begin{pmatrix} -15 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Solving Simultaneous Equations

General reduction scheme for Gaussian Elimination:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix}$$

Solving Simultaneous Equations

General reduction scheme for Gaussian Elimination:

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} a_{kj}^{(k-1)}, i, j = k+1, \ldots, n$$

$$b_{i}^{(k)} = b_{i}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} b_{k}^{(k-1)}, i = k+1, \ldots, n$$

$$\therefore x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}, i = n-1, n-2, \ldots, 1$$

Solving Simultaneous Equations

Example:

$$2x_1 + 3x_2 - x_3 + 4x_4 = 19$$
$$x_1 - x_2 + 2x_3 - 2x_4 = 3$$
$$4x_1 + 2x_2 - 3x_3 - x_4 = 15$$
$$3x_1 + 4x_2 - 2x_3 + x_4 = 21$$

Solve for x_1, x_2, x_3 and x_4.

Gaussian Elimination

$$\begin{pmatrix} 2 & 3 & -1 & 4 \\ 1 & -1 & 2 & -2 \\ 4 & 2 & -3 & -1 \\ 3 & 4 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 19 \\ 3 \\ 15 \\ 21 \end{pmatrix}$$

Solving Simultaneous Equations

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Solve for x_1, x_2, x_3 and x_4.

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} a_{kj}^{(k-1)}, i, j = k+1, \ldots, n$$

$$b_i^{(k)} = b_i^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} b_k^{(k-1)}, i = k+1, \ldots, n$$

Gaussian Elimination

Step 1: k=1

$$a_{ij}^{(1)} = a_{ij}^{(0)} - \frac{a_{i1}^{(0)}}{a_{11}^{(0)}} a_{1j}^{(0)}, i, j = 2, 3, 4$$

$$b_i^{(1)} = b_i^{(0)} - \frac{a_{i1}^{(0)}}{a_{11}^{(0)}} b_1^{(0)}, i = 2, 3, 4$$

- Pivot row 1 (k=1)
- Eliminate x_1 from row 2,3 and 4

Solving Simultaneous Equations

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Solve for x_1, x_2, x_3 and x_4.

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} a_{kj}^{(k-1)}, i, j = k+1, \ldots, n$$

$$b_i^{(k)} = b_i^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} b_k^{(k-1)}, i = k+1, \ldots, n$$

Gaussian Elimination

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$$a_{ij}^{(1)} = a_{ij}^{(0)} - \frac{a_{i1}^{(0)}}{a_{11}^{(0)}} a_{1j}^{(0)}, i, j = 2,3,4$$

$$b_i^{(1)} = b_i^{(0)} - \frac{a_{i1}^{(0)}}{a_{11}^{(0)}} b_1^{(0)}, i = 2,3,4$$

$$\begin{pmatrix} 2 & 3 & -1 & 4 \\ 0 & -2.5 & 2.5 & -4 \\ 0 & -4 & -1 & -9 \\ 0 & -0.5 & -0.5 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 19 \\ -6.5 \\ -23 \\ -7.5 \end{pmatrix}$$

Solving Simultaneous Equations

Example:

$$2x_1 + 3x_2 - x_3 + 4x_4 = 19$$
$$x_1 - x_2 + 2x_3 - 2x_4 = 3$$
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Solve for x_1, x_2, x_3 and x_4.

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} a_{kj}^{(k-1)}, i, j = k+1, \ldots, n$$

$$b_i^{(k)} = b_i^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} b_k^{(k-1)}, i = k+1, \ldots, n$$

Gaussian Elimination

Step 2: k=2

$$a_{ij}^{(2)} = a_{ij}^{(1)} - \frac{a_{i2}^{(1)}}{a_{22}^{(1)}} a_{2j}^{(1)}, i, j = 3, 4$$

$$b_i^{(2)} = b_i^{(1)} - \frac{a_{i2}^{(1)}}{a_{22}^{(1)}} b_2^{(1)}, i = 3, 4$$

$$\begin{pmatrix} 2 & 3 & -1 & 4 \\ 0 & -2.5 & 2.5 & -4 \\ 0 & 0 & -5 & -2.6 \\ 0 & 0 & -1 & -4.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 19 \\ -6.5 \\ -12.6 \\ -6.2 \end{pmatrix}$$

Solving Simultaneous Equations

Example:

$$2x_1 + 3x_2 - x_3 + 4x_4 = 19$$
$$x_1 - x_2 + 2x_3 - 2x_4 = 3$$
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Solve for x_1, x_2, x_3 and x_4.

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} a_{kj}^{(k-1)}, i, j = k+1, \ldots, n$$

$$b_{i}^{(k)} = b_{i}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} b_{k}^{(k-1)}, i = k+1, \ldots, n$$

Gaussian Elimination

Step 3: k=3

$$a_{ij}^{(3)} = a_{ij}^{(2)} - \frac{a_{i3}^{(2)}}{a_{33}^{(2)}} a_{3j}^{(2)}, i, j = 4$$

$$b_{i}^{(3)} = b_{i}^{(2)} - \frac{a_{i3}^{(2)}}{a_{33}^{(2)}} b_{2}^{(2)}, i = 4$$

$$\begin{pmatrix} 2 & 3 & -1 & 4 \\ 0 & -2.5 & 2.5 & -4 \\ 0 & 0 & -5 & -2.6 \\ 0 & 0 & 0 & -3.68 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 19 \\ -6.5 \\ -12.6 \\ -3.68 \end{pmatrix}$$

Solving Simultaneous Equations

Example:

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Solve for x_1, x_2, x_3 and x_4.

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}}, i = n-1, n-2, \ldots, 1$$

Gaussian Elimination

Step 4: calculate x_1, x_2, x_3, x_4

$$\begin{pmatrix} 2 & 3 & -1 & 4 \\ 0 & -2.5 & 2.5 & -4 \\ 0 & 0 & -5 & -2.6 \\ 0 & 0 & 0 & -3.68 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 19 \\ -6.5 \\ -12.6 \\ -3.68 \end{pmatrix}$$

Solving Simultaneous Equations

Example:

$$2x_1 + 3x_2 - x_3 + 4x_4 = 19$$
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Solve for x_1, x_2, x_3 and x_4.

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}, i = n-1, n-2, \ldots, 1$$

Gaussian Elimination

Step 4: calculate x_1, x_2, x_3, x_4

$$\therefore x_4 = 1$$

$$\therefore x_3 = \frac{b_3 - \sum_{j=4}^{4} a_{3j}x_j}{a_{33}} = \frac{-12.6 - (-2.6 \times 1)}{-5} = 2$$

$$\therefore x_2 = \frac{b_2 - \sum_{j=3}^{4} a_{2j}x_j}{a_{22}} = \frac{-6.5 - (2.5 \times 2 - 4 \times 1)}{-2.5} = 3$$

$$\therefore x_1 = \frac{b_2 - \sum_{j=2}^{4} a_{2j}x_j}{a_{11}} = \frac{19 - (3 \times 3 - 1 \times 2 + 4 \times 1)}{2} = 4$$

Solving Simultaneous Equations

Generalisation for Gaussian Elimination method seems to be tedious

But the formulation allows for ease implementation in programming environment

Solving Simultaneous Equations

Why do we need to know how to solve simultaneous equations for FEM?

!! This forms the basis in solving the stiffness matrix for FEM !!

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You are expected to be able:

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