SSCE1693 ENGINEERING MATHEMATICS

CHAPTER 5:

SERIES

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Revision

Sequence

What is a sequence? It is a set of numbers which are written in some particular order

$$u_1, u_2, \ldots, u_n.$$

We sometimes write u_1 for the first term of the sequence, u_2 for the second term and so on. We write the n^{th} term as u_n.

Examples:

1, 3, 5, 9. – finite sequence

1, 2, 3, 4, 5, …, n – finite sequence

1, 1, 2, 3, 5, 8, … - infinite sequence

Two types of sequence:

1. Geometric Sequence
2. Arithmetic Sequence

Arithmetic Sequence

- Arithmetic sequence is a sequence where each new term after the first is obtained by **adding** a constant *d*, called the **common difference** to the preceding term.
- If the term of the sequence is *a*, then the AS is

  $$a, a+d, a+2d, a+3d, ...$$

  where the *n*-th term is $a+(n-1)d$.
- Examples:

  8, 5, 2, -1, -4, .. $(a=8, d=-3)$

Geometric Sequence

- GS is a sequence where each new term after the first is obtained by **multiplying** the preceding term by a constant *r*, called the **common ratio.**
- If the first term of the sequence is *a*, then the GS is

  $$a, ar, ar^2, ar^3, \ldots$$

  where the *n*-th term is ar^{n-1}.
- Examples:

  2, 6, 18, 54, .. $(a = 2, r = 3)$

  1, -2, 4, -8 $(a = 1, r = -2)$

5.1 Series

For example, suppose we have the sequence

$$u_1, u_2, \cdots, u_n.$$

The series we obtain from this is

$$u_1 + u_2 + \cdots + u_n.$$

and we write S_n for the sum of these *n* terms.

For example, let us consider the sequence of numbers

$$1, 2, 3, 4, 5, 6, \ldots, n.$$

Then,

$$S_1 = 1$$
$$S_2 = 1 + 2 = 3$$
$$S_3 = 1 + 2 + 3 = 6$$

The difference between the sum of two consecutive partial terms, $S_n - S_{n-1}$, is the n^{th} term of the series.

$$u_n = S_n - S_{n-1}$$

- If the sum of the terms ends after a few terms, then the series is called **finite series**.
- If the sum of the series does not end, then the series is called **infinite series**.

Example 5.1:

Find the 4^{th} term and 5^{th} term of the sequence

1, 4, 7, ….

Hence, find S_4 and S_5 of the series 1, 4, 7, …. .

Example 5.2:

The sum of the first *n* terms of the series is given by

$$S_n = \frac{1}{4}(5n^2 + 11n)$$

a) Find the first three terms, and

b) The *n*-th term of the series.

Summation Notation, $\sum$.

$\sum$ (read as sigma) is used to represent the sum of the series. In general,

$$S_n = u_1 + u_2 + \cdots + u_n = \sum_{r=1}^{n} u_r. \text{ (finite)}$$

$$S_\infty = u_1 + u_2 + u_3 + \cdots = \sum_{i=1}^{\infty} u_i. \text{ (infinite)}$$

Example 5.3:

Find the *r*-th term of the following series. Hence, express the series using $\sum$ notation.

a) $2+3+4+\ldots$, to 10 terms.

b) $-3+9-27+\ldots$, until 30 terms.

5.2 The Sum of a Series

5.2.1 Sum of Power of '*n*' Positive Integers

$$\sum_{r=1}^{n} r = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\sum_{r=1}^{n} r^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = 1^3+2^3+3^3+\cdots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Example 5.4:

Evaluate $\sum_{r=1}^{20} r^2$ and $\sum_{r=1}^{25} r^3$.

Example 5.5:

Evaluate $\sum_{r=1}^{10}(2r-1)^2$.

Example 5.6:

Find the sum for each of the following series:

(a) $2^2+4^2+6^2+\cdots+(2n)^2$

(b) $1\cdot 3+4\cdot 5+7\cdot 7+\cdots$ to 30 terms

5.2.2 Sum of Series of Partial Fraction

In this section we shall discuss terms with partial fractions such as

$$\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+...$$

We are not able to calculate the sum of the series by using the available formula (so far), but with the help of partial fraction method, we can solve the problem.

Example 5.7:

Find the sum of the first *n* terms of the series

$$\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+...$$

The above problem requires quite a long solution. However, in the next sub-topic, we will see a different approach to solve the same problem. We called the approach, a **difference method.**

5.3 Test of Convergence

5.3.1 Divergence Test

Note: This test only determines the divergence of a series.

Example 5.8:

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Example 5.9:

Use Divergence Test to determine whether $\sum_{r=1}^{\infty} \frac{r}{\ln r}$ diverges or not.

5.3.2 The Integral Test

Note: Use this test when $f(x)$ is easy to integrate.

Example 5.10:

Use the integral test to determine whether the following series converges or diverges.

(a) $\displaystyle\sum_{r=2}^{\infty} \frac{1}{r \ln r}$.

(b) $\displaystyle\sum_{r=1}^{\infty} \frac{r}{\sqrt{r^2+4}}$.

5.3.3 Ratio Test

Example 5.11:

Use the Ratio Test to determine whether the following series converges or diverges.

(a) $\displaystyle\sum_{r=1}^{\infty} \frac{r^2}{4^r}$.

(b) $\displaystyle\sum_{r=1}^{\infty} r\, e^{-r}$.

5.4 Power Series

5.4.1 Expansion of Exponent Function

The power series of the exponent function can be written as

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots$$

The expansion is true for all values of x. In general,

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n .$$

Example 5.12:

Given

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots + \frac{1}{n!}x^n + \ldots$$

Write down the first five terms of the expansion of the following functions

(a) e^{2x} (b) e^{x-1}

Example 5.13:

Write down the first five terms on the expansion of the function, $(1+x)^2 e^{-x}$ in the form of power series.

5.4.2 Expansion of Logarithmic Function

The expansion of logarithmic function can be written as

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 - \dots$$

The series converges for $-1 < x \leq 1$. Thus the series $\ln(1 + x)$ is valid for $-1 < x \leq 1$.

By assuming x with $-x$, we obtain

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \frac{1}{6}x^6 - \frac{1}{7}x^7 - \dots$$

Thus, this result is true for $-1 < -x \leq 1$ or $-1 \leq x < 1$.

Example 5.14:

Write down the first five terms of the expansion of the following functions

(a) $\ln(1+3x)$

(b) $3\ln(1-2x^2)(1+3x)$

Example 5.15:

Find the first four terms of the expansion of the function, $(1+x)^2\ln(1+2x)^3$.

5.4.3 Expansion of Trigonometric Function

The power series for trigonometric functions can be written as

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

Both series are valid for all values of x.

Example 5.16:

Given

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

Find the expansion of $\cos(2x)$ and $\cos(3x)$. Hence, by using an appropriate trigonometric identity find the first four terms of the expansion of the following functions:

(a) $\sin^2(x)$

(b) $\cos^3(x)$

5.5 Taylor and the Maclaurin Series

If $f(x)$ has a derivatives of all orders at $x=a$, then we call the series as Taylor's series for $f(x)$ about $x=a$, and is given b

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\cdots$$
$$+\frac{(x-a)^r}{r!}f^r(a),$$

or

$$f(x+a)=f(a)+x\,f'(a)+\frac{x^2}{2!}f''(a)+\frac{x^3}{3!}f'''(a)+\cdots+\frac{x^r}{r!}f^r(a).$$

In the special case where $a=0$, this series becomes the Maclaurin series for $f(x)$ and is given by

$$f(x)=f(0)+x\,f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots+\frac{x^r}{r!}f^r(0).$$

Example 5.17:

Obtain the Taylor series for $f(x) = 3x^2 - 6x + 5$ around the point $x = 1$.

Ans: 2 + 3(x-1)^2

Example 5.18:

Obtain Maclaurin series expansion for the first four terms of e^x and five terms of $\sin x$. Hence, deduct that Maclaurin series for $e^x \sin x$ is given by

$$x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \ldots$$

Example 5.19:

Use Taylor's theorem to obtain a series expansion of first five derivatives for $\cos\left(x + \frac{\pi}{3}\right)$. Hence find $\cos 62^0$ correct to 4 dcp.

Ans: 0.4695

Example 5.20:

If $y = \ln\cos x$, show that

$$\frac{d^2y}{dx^2} + 1 + \left(\frac{dy}{dx}\right)^2 = 0$$

Hence, by differentiating the above expression several times, obtain the Maclaurin's series of $y = \ln\cos x$ in the ascending power of *x* up to the term containing x^4.

5.5.1 *Finding Limits with Taylor Series and Maclaurin Series.*

Example 5.21:

Find $\lim_{x\to 0}\dfrac{e^x - 1 - x}{x^2}$. Ans: ½

Example 5.22:

$\lim_{x\to 0}\dfrac{x^2 + 2\cos x - 2}{3x^4}$. Ans: 1/36

5.5.2 Evaluating Definite Integrals with Taylor Series and Maclaurin Series.

Example 5.23:

Use the first 6 terms of the Maclaurin series to approximate the following definite integral.

a) $\int_0^1 e^{-x^2}\,dx$ Ans: 0.747

b) $\int_0^1 x\cos(x^3)\,dx$ Ans: 0.440