

# Mathematical Methods III

## SSCM 2043

### Chapter 1

This chapter discuss about even, odd functions and Fourier series

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# Periodic Functions

Any function that satisfies

$$f(t) = f(t + T)$$

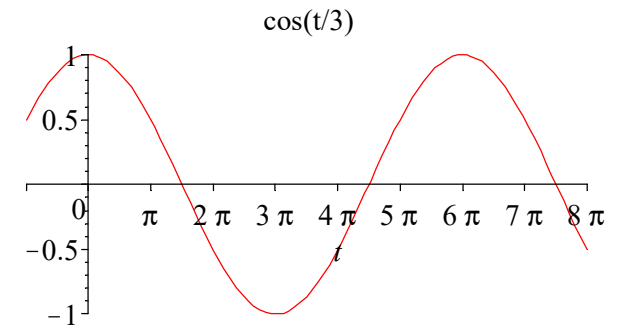
where  $T$  is a constant and is called the *period* of the function.

$$f(t) = \cos \frac{t}{3} + \cos \frac{t}{4}$$

Example: Find its period

$$f(t) = f(t + T)$$

→  $\cos \frac{t}{3} + \cos \frac{t}{4} = \cos \frac{1}{3}(t + T) + \cos \frac{1}{4}(t + T)$



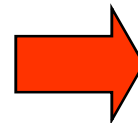
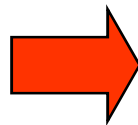
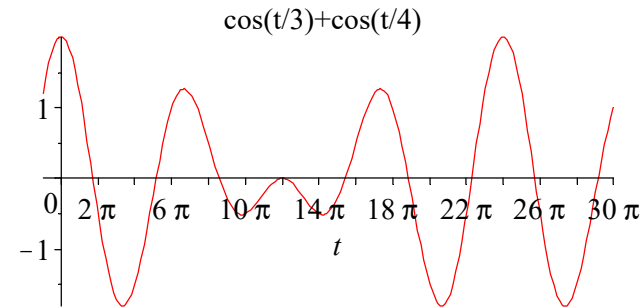
Fact:  $\cos \theta = \cos(\theta + 2m\pi)$

$$\frac{T}{3} = 2m\pi$$

$$T = 6m\pi$$

$$\frac{T}{4} = 2n\pi$$

$$T = 8n\pi$$



$$T = 24\pi$$

smallest  $T$   
(for both)

# Periodic Functions

Example:

$$f(t) = \cos \omega_1 t + \cos \omega_2 t$$

Find its period.

$$f(t) = f(t + T) \implies \cos \omega_1 t + \cos \omega_2 t = \cos \omega_1 (t + T) + \cos \omega_2 (t + T)$$

$$\omega_1 T = 2m\pi$$

$$\omega_2 T = 2n\pi$$

$$\implies \frac{\omega_1}{\omega_2} = \frac{m}{n}$$

$$\implies \frac{\omega_1}{\omega_2}$$

→ must be a rational number

$$f(t) = \cos 5t + \cos(5 + \pi)t$$



Is this function a periodic function?

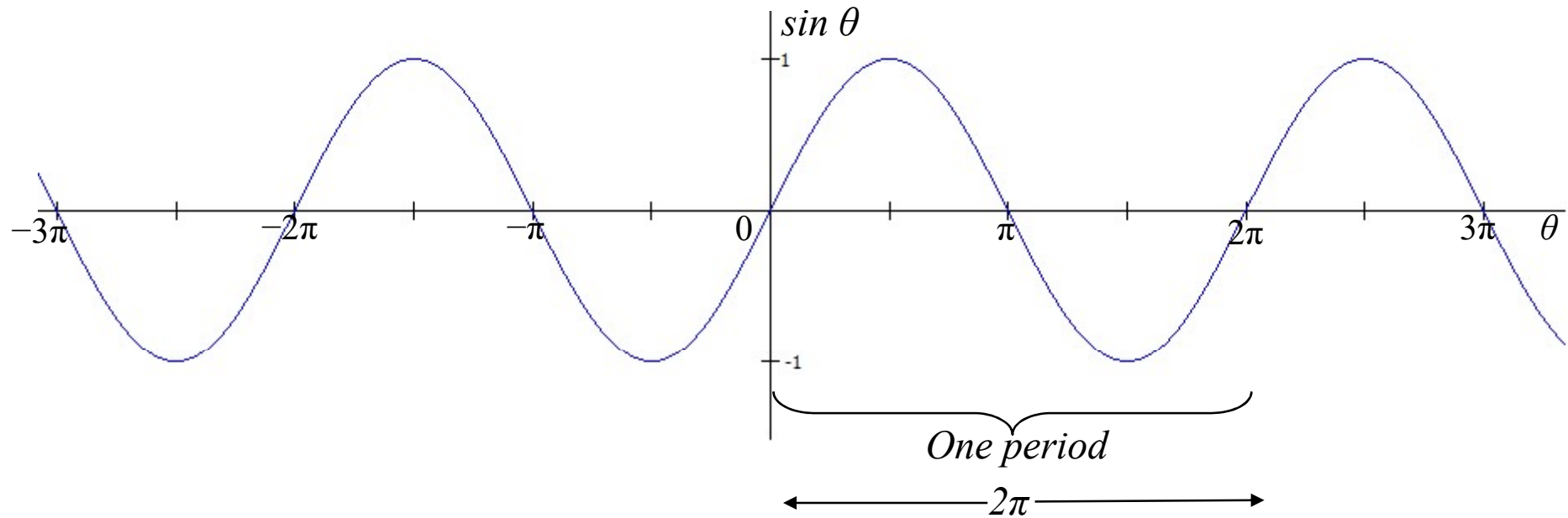
$$\frac{\omega_1}{\omega_2} = \frac{5}{5 + \pi}$$

→ not a rational number

$$\cos \omega(t + T) = \cos \omega t \cos \omega T - \sin \omega t \sin \omega T$$

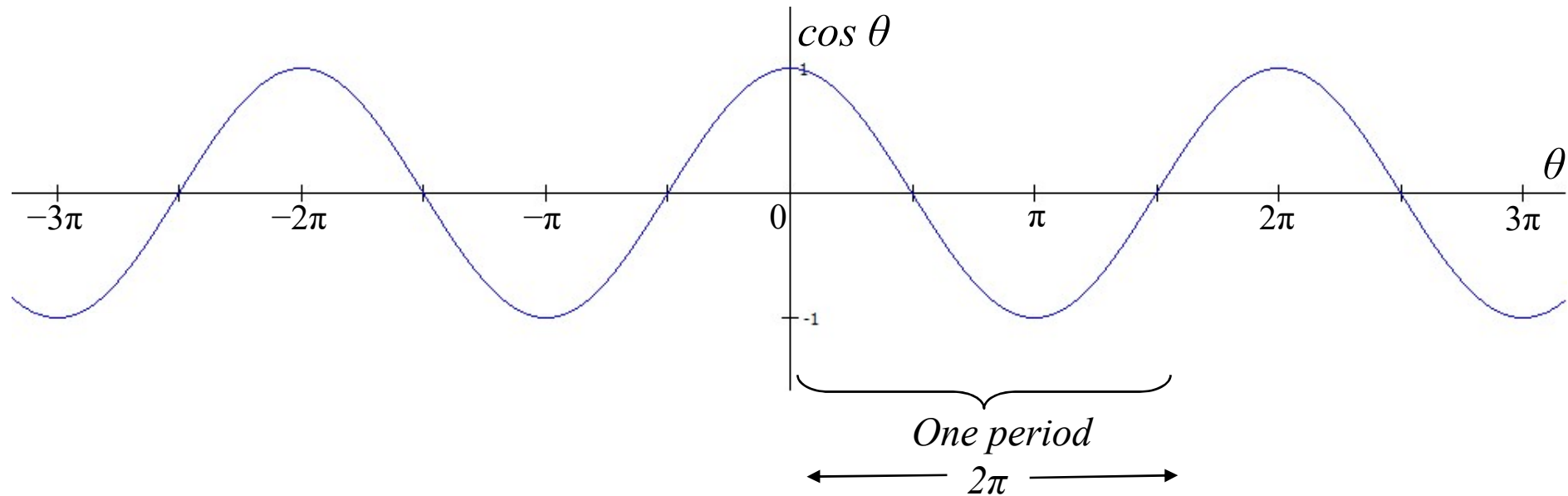
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

# Odd Functions



$\sin \theta$  is an odd function; it is symmetric w.r.t. the origin. **Definition:**  $f(-t) = -f(t)$   
 $\sin(-\theta) = -\sin(\theta)$

# Even Functions



$\cos \theta$  is an **even** function; it is symmetric w.r.t. the y-axis. : **Definition:**  $f(-t) = f(t)$   
 $\cos(-\theta) = \cos(\theta)$

## Fourier Series

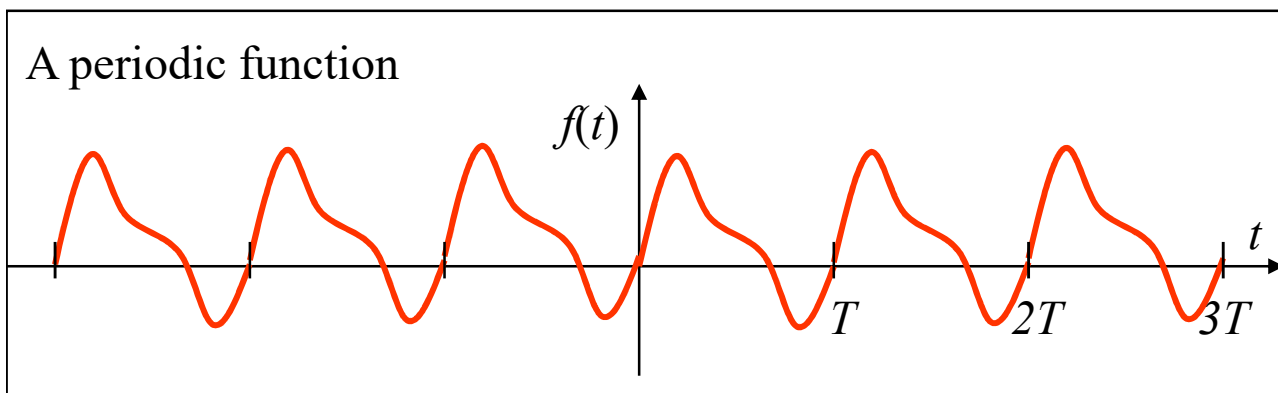
$$f(t) = \underbrace{\frac{a_0}{2}}_{\text{DC Part}} + \underbrace{\sum_{n=1}^{\infty} a_n \cos \frac{2\pi n t}{T}}_{\text{Even Part}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin \frac{2\pi n t}{T}}_{\text{Odd Part}}$$

$T$  is a period of all the above signals

Let  $\omega = 2\pi/T$ ;  $T = 2l$

Note: Average component is similar to DC component in signal

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$



# Orthogonal Functions

- A set of functions  $\{\psi_k\}$  is *orthogonal* on an interval  $a < t < b$  if it satisfies

$$\int_a^b \psi_m(t)\psi_n(t)dt = \begin{cases} 0 & m \neq n \\ c (\neq 0) & m = n \end{cases}$$

# Fourier Series

## Orthogonality in Sinusoidal Functions

Define  $\omega = 2\pi/T$ .  $m$  and  $n$  are integer

$$\int_{-T/2}^{T/2} \cos(m\omega t) dt = 0, \quad m \neq 0$$

$$\int_{-T/2}^{T/2} \sin(m\omega t) dt = 0, \quad m \neq 0$$

$$\int_{-T/2}^{T/2} \cos(m\omega t) \cos(n\omega t) dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega t) \sin(n\omega t) dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega t) \cos(n\omega t) dt = 0, \quad \text{for all } m \text{ and } n$$

} Try this !



# Fourier Series

## Orthogonality in Sinusoidal Functions

### Proof

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\omega = 2\pi/T; \quad T = 2l$$

$$\int_{-T/2}^{T/2} \cos(m\omega t) \cos(n\omega t) dt \quad m \neq n$$

$$= \frac{1}{2} \int_{-T/2}^{T/2} \cos[(m+n)\omega t] dt + \frac{1}{2} \int_{-T/2}^{T/2} \cos[(m-n)\omega t] dt$$

$$= \frac{1}{2} \frac{1}{(m+n)\omega} \sin[(m+n)\omega t] \Big|_{-T/2}^{T/2} + \frac{1}{2} \frac{1}{(m-n)\omega} \sin[(m-n)\omega t] \Big|_{-T/2}^{T/2}$$

$$= \frac{1}{2} \frac{1}{(m+n)\omega} \underbrace{2 \sin[(m+n)\pi]}_0 + \frac{1}{2} \frac{1}{(m-n)\omega} \underbrace{2 \sin[(m-n)\pi]}_0 = 0$$

$$\left\{ \begin{array}{l} 1, \\ \cos \omega t, \cos 2\omega t, \cos 3\omega t, \dots \\ \sin \omega t, \sin 2\omega t, \sin 3\omega t, \dots \end{array} \right\}$$

**is orthogonal set.**

# Fourier series: Decomposition

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) \quad \begin{array}{l} \omega = 2\pi/T \\ T = 2l \end{array}$$

$$a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) dt = \frac{1}{l} \int_{-l}^l f(t) dt \quad a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega t dt = \frac{1}{l} \int_{-l}^l f(t) \cos n\omega t dt, \quad n = 1, 2, \dots$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega t dt = \frac{1}{l} \int_{-l}^l f(t) \sin n\omega t dt, \quad n = 1, 2, \dots$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{l}\right)$$

$$\int_{-T/2}^{T/2} \cos(m\omega t) dt = 0, \quad m \neq 0$$

$$\int_{-T/2}^{T/2} \cos(m\omega t) \cos(n\omega t) dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega t) dt = 0, \quad m \neq 0$$

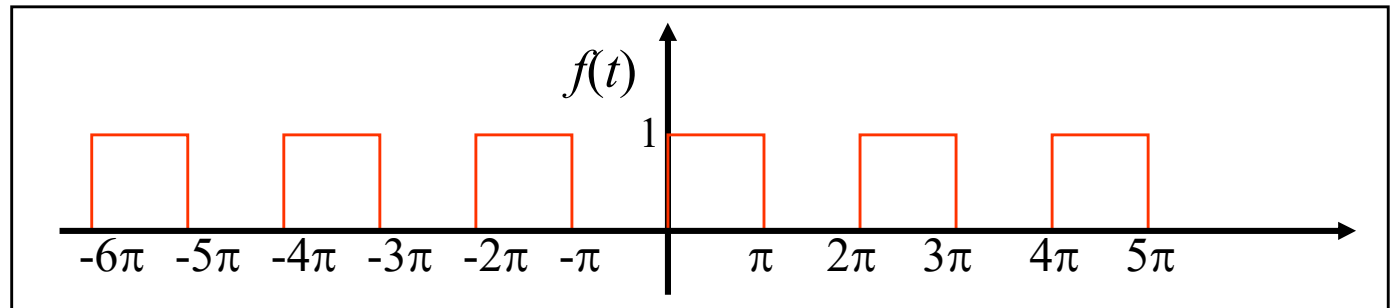
$$\int_{-T/2}^{T/2} \sin(m\omega t) \cos(n\omega t) dt = 0, \quad \text{for all } m \text{ and } n$$

$$\int_{-T/2}^{T/2} \sin(m\omega t) \sin(n\omega t) dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

# Fourier Series: Example (Square Wave)

$$f(t) = \begin{cases} 1, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

$$f(t) = f(t + 2\pi)$$



Period

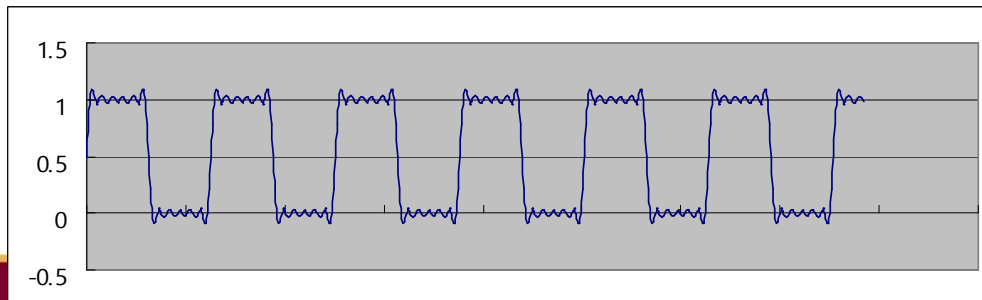
$$T = 2l = 2\pi$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$a_0 = \frac{2}{2\pi} \int_0^{\pi} 1 dt = 1$$

$$a_n = \frac{2}{2\pi} \int_0^{\pi} \cos ntdt = \frac{1}{n\pi} \sin nt \Big|_0^{\pi} = 0 \quad n = 1, 2, \dots$$

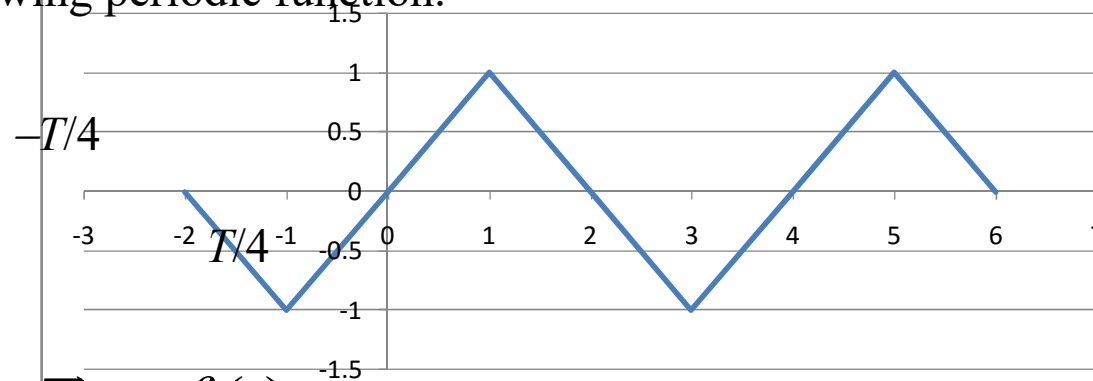
$$b_n = \frac{2}{2\pi} \int_0^{\pi} \sin ntdt = -\frac{1}{n\pi} \cos nt \Big|_0^{\pi} = -\frac{1}{n\pi} (\cos n\pi - 1) = \begin{cases} 2/n\pi & n = 1, 3, 5, \dots \\ 0 & n = 2, 4, 6, \dots \end{cases}$$



$$f(t) = \frac{1}{2} + \frac{2}{\pi} \left( \sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \dots \right)$$

# Fourier Series

Example Find the Fourier series of the following periodic function.



$$f(t) = \begin{cases} t, & -\frac{T}{4} \leq t \leq \frac{T}{4} \\ -t + \frac{T}{2}, & \frac{T}{4} \leq t \leq \frac{3T}{4} \end{cases}$$

$$f(t+T) = f(t)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

This is an odd function. Therefore,  $a_n=0$

$$\omega = 2\pi/T.$$

$$T = 2l$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin\left(\frac{2\pi n}{T} t\right) dt$$

$$= \frac{4}{T} \int_0^{\frac{T}{2}} f(t) \sin\left(\frac{2\pi n}{T} t\right) dt$$

$$a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) dt = \frac{1}{l} \int_{-l}^l f(t) dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega t dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega t dt$$

# Fourier Series

$$b_n = \frac{4}{T} \int_0^{\frac{T}{4}} t \sin\left(\frac{2\pi n}{T} t\right) dt + \frac{4}{T} \int_{\frac{T}{4}}^{\frac{T}{2}} \left(-t + \frac{T}{2}\right) \sin\left(\frac{2\pi n}{T} t\right) dt$$

$b_n = 0$ , when  $n$  is even.

Therefore, the Fourier series is

$$f(t) = \frac{2T}{\pi^2} \left[ \sin\left(\frac{2\pi}{T} t\right) - \frac{1}{3^2} \sin\left(\frac{6\pi}{T} t\right) + \frac{1}{5^2} \sin\left(\frac{10\pi}{T} t\right) - \dots \right]$$

$$b_n = \frac{4}{T} \left[ 2 \left(\frac{T}{2\pi n}\right)^2 \sin\left(\frac{n\pi}{2}\right) \right] = \frac{2T}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right)$$

Use integration by parts.

Example of integration by parts.

$$\int_{x=a}^{x=b} x^2 \sin(nx) dx$$

differentiate	Integrate
$x^2$ +1	$\sin nx$
$2x$ -1	$\left(-\frac{1}{n}\right) \cos nx$
$2$ +1	$\left(-\frac{1}{n^2}\right) \sin nx$
$0$	$\left(\frac{1}{n^3}\right) \cos nx$

$$\int_{x=a}^{x=b} x^2 \sin(nx) dx = \left[ x^2 \left(-\frac{1}{n}\right) \cos nx + 2x(-1) \left(-\frac{1}{n^2}\right) \sin nx + 2 \left(\frac{1}{n^3}\right) \cos nx \right]_{x=a}^{x=b}$$

# Fourier Series Harmonics

Define  $\omega = 2\pi f = \frac{2\pi}{T}$  called the *fundamental angular frequency*.

Define  $\omega_n = n\omega$  called the *n-th harmonic* of the periodic function.

Where  $\omega = \omega_0$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t$$

# Complex Form of the Fourier Series

$$e^{jn\omega t} = \cos n\omega t + j \sin n\omega t \quad e^{-jn\omega t} = \cos n\omega t - j \sin n\omega t$$

$$\cos n\omega t = \frac{1}{2} \left( e^{jn\omega t} + e^{-jn\omega t} \right) \quad \sin n\omega t = \frac{1}{2j} \left( e^{jn\omega t} - e^{-jn\omega t} \right) = -\frac{j}{2} \left( e^{jn\omega t} - e^{-jn\omega t} \right)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$= \frac{a_0}{2} + \frac{1}{2} \sum_{n=1}^{\infty} a_n \left( e^{jn\omega t} + e^{-jn\omega t} \right) - \frac{j}{2} \sum_{n=1}^{\infty} b_n \left( e^{jn\omega t} - e^{-jn\omega t} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ \frac{1}{2} (a_n - jb_n) e^{jn\omega t} + \frac{1}{2} (a_n + jb_n) e^{-jn\omega t} \right]$$

$$= c_0 + \sum_{n=1}^{\infty} \left[ c_n e^{jn\omega t} + c_{-n} e^{-jn\omega t} \right]$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega t} + \sum_{n=-\infty}^{-1} c_n e^{jn\omega t}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

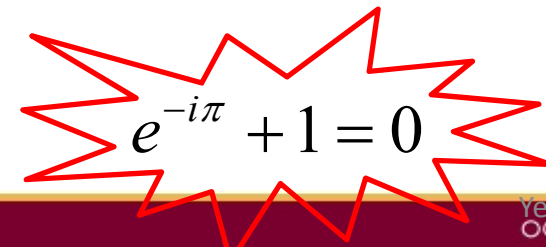
$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2} (a_n - jb_n)$$

$$c_{-n} = \frac{1}{2} (a_n + jb_n)$$

$$\omega = 2\pi/T$$

$$T = 2l$$



$$e^{-i\pi} + 1 = 0$$

## Complex Form of the Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

$$\omega = 2\pi/T$$

$$T = 2l$$

$$c_0 = \frac{a_0}{2} = \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega t} dt$$

$$c_n = \frac{1}{2} (a_n - jb_n)$$

$$= \frac{1}{T} \left[ \int_{-T/2}^{T/2} f(t) \cos n\omega t dt - j \int_{-T/2}^{T/2} f(t) \sin n\omega t dt \right]$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} f(t) (\cos n\omega t - j \sin n\omega t) dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega t} dt$$

$$c_{-n} = \frac{1}{2} (a_n + jb_n) = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{jn\omega t} dt$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2} (a_n - jb_n)$$

$$c_{-n} = \frac{1}{2} (a_n + jb_n)$$

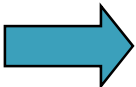


## Complex Form of the Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega t} dt$$

If  $f(t)$  is real,

  $c_{-n} = c_n^*$

$$c_n = |c_n| e^{j\phi_n}, \quad c_{-n} = c_n^* = |c_n| e^{-j\phi_n}$$

$$|c_n| = |c_{-n}| = \frac{1}{2} \sqrt{a_n^2 + b_n^2}$$

$$\phi_n = \tan^{-1} \left( -\frac{b_n}{a_n} \right)$$

$$n = \pm 1, \pm 2, \pm 3, \dots$$

$$c_0 = \frac{1}{2} a_0$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2} (a_n - jb_n)$$

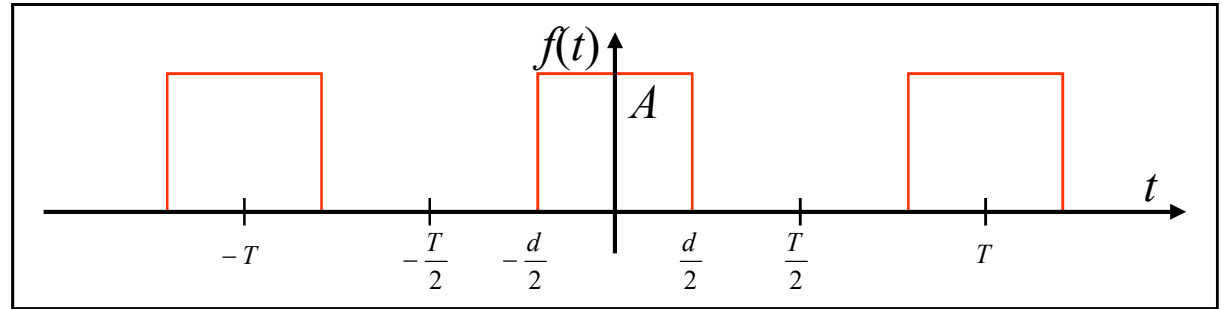
$$c_{-n} = \frac{1}{2} (a_n + jb_n)$$

# Complex Form of the Fourier Series

## Example

$$f(t) = \begin{cases} 0, & -T/2 < t < -d/2 \\ A, & -d/2 < t < d/2 \\ 0, & d/2 < t < T/2 \end{cases}$$

$$f(t) = f(t+T)$$



$$c_n = \frac{A}{T} \int_{-d/2}^{d/2} e^{-jn\omega t} dt$$

$$= \frac{A}{T} \frac{1}{-jn\omega} e^{-jn\omega t} \Big|_{-d/2}^{d/2}$$

$$= \frac{A}{T} \left( \frac{1}{-jn\omega} e^{-jn\omega d/2} - \frac{1}{-jn\omega} e^{jn\omega d/2} \right)$$

$$= \frac{A}{T} \frac{1}{-jn\omega} (-2j \sin n\omega d/2)$$

$$= \frac{A}{T} \frac{1}{\frac{1}{2}n\omega} \sin n\omega d/2 = \frac{Ad}{T} \frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)}$$

$$\cos n\omega t = \frac{1}{2} (e^{jn\omega t} + e^{-jn\omega t})$$

$$\sin n\omega t = \frac{1}{2j} (e^{jn\omega t} - e^{-jn\omega t}) = -\frac{j}{2} (e^{jn\omega t} - e^{-jn\omega t})$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega t} dt$$

## SOLVED PROBLEM

1. Find the complex form of the Fourier series of the periodic function

$$f(x) = \begin{cases} 0, & 0 < x < l \\ \alpha, & l < x < 2l \end{cases}$$

## SOLUTION

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in \pi x}{l}}$$

$$c_n = \frac{1}{2l} \int_0^{2l} f(x) e^{-\frac{in \pi x}{l}} dx$$

$$c_n = \frac{1}{2l} \left[ \int_0^l 0 \left( e^{-\frac{in \pi x}{l}} \right) dx + \int_l^{2l} \alpha e^{-\frac{in \pi x}{l}} dx \right]$$

$$= \frac{\alpha}{2l} \left( -\frac{l}{in \pi} e^{-\frac{in \pi x}{l}} \right)_l^{2l}$$

$$= -\frac{\alpha}{2n \pi i} \left[ e^{-2in \pi} - e^{-ni \pi} \right]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n \omega t + \sum_{n=1}^{\infty} b_n \sin n \omega t$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn \omega t}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn \omega t} dt$$

$$\omega = 2\pi/T.$$

$$T = 2l$$

$$c_n = -\frac{\alpha}{2n\pi i} [1 - (-1)^n]$$

$$= \frac{\alpha i}{2n\pi} [1 - (-1)^n] \quad \text{If } n \neq 0$$

If  $n=0$

$$c_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega t} dt$$

$$= \frac{1}{2l} \int_l^{2l} \alpha e^{-0} dt = \frac{\alpha}{2}$$

$$f(x) = c_0 + \sum_{n=-\infty, n \neq 0}^{\infty} \frac{\alpha i}{2n\pi} (1 - (-1)^n) e^{\frac{in\pi x}{l}}$$

$$\cos n\omega t = \frac{1}{2} (e^{jn\omega t} + e^{-jn\omega t})$$

$$\sin n\omega t = -\frac{j}{2} (e^{jn\omega t} - e^{-jn\omega t})$$

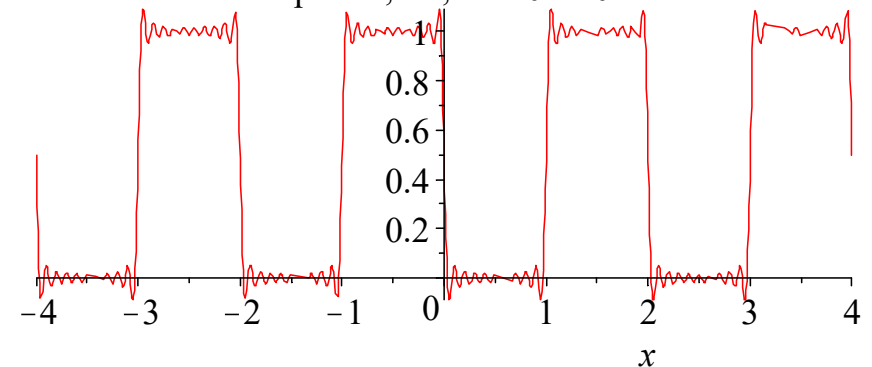
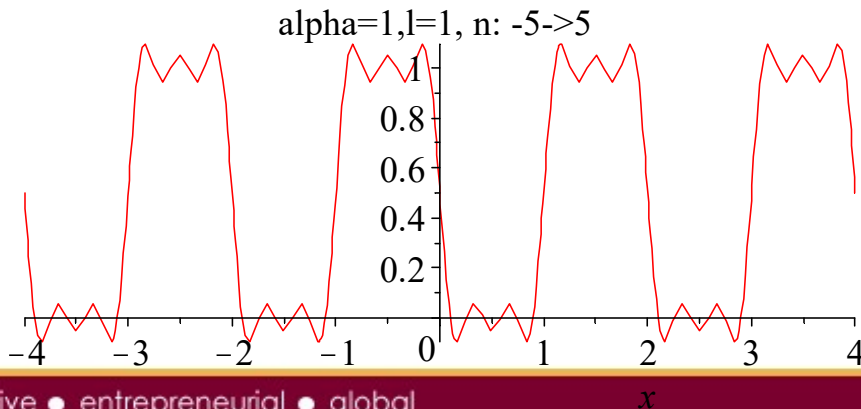
$$f(x) = c_0 + \sum_{n=-\infty, n \neq 0}^{\infty} \frac{\alpha i}{2n\pi} (1 - (-1)^n) \left\{ \cos\left(\frac{n\pi x}{l}\right) + i \sin\left(\frac{n\pi x}{l}\right) \right\}$$

$$= \frac{\alpha}{2} + \sum_{n=-\infty, n \neq 0}^{\infty} \frac{i\alpha}{2n\pi} (1 - (-1)^n) \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=-\infty, n \neq 0}^{\infty} \frac{-\alpha}{2n\pi} (1 - (-1)^n) \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{\alpha}{2} + \sum_{k=1}^{\infty} \frac{\alpha}{2k\pi} (1 - (-1)^{-k}) \sin\left(-\frac{k\pi x}{l}\right) + \sum_{n=1}^{\infty} \frac{-\alpha}{2n\pi} (1 - (-1)^n) \sin\left(\frac{n\pi x}{l}\right) = \frac{\alpha}{2} + \sum_{n=1}^{\infty} \frac{-\alpha}{n\pi} (1 - (-1)^n) \sin\left(\frac{n\pi x}{l}\right)$$

If  $k = -n$

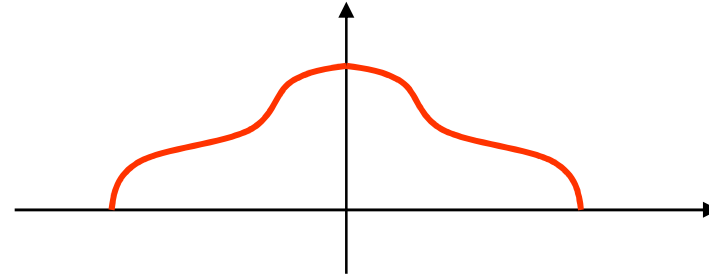
alpha=1, l=1, n: -20->20



# Symmetry property

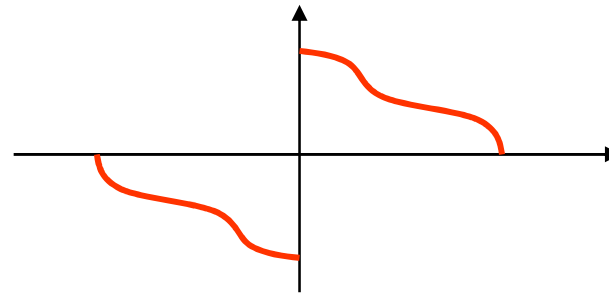
## Even Functions

$$f(t) = f(-t)$$



## Odd Functions

$$f(t) = -f(-t)$$



- Any function  $f(t)$  can be expressed as the sum of an even function  $f_e(t)$  and an odd function  $f_o(t)$ .

$$f(t) = f_e(t) + f_o(t)$$

$$f_e(t) = \frac{1}{2}[f(t) + f(-t)] \quad \text{Even Part}$$

$$f_o(t) = \frac{1}{2}[f(t) - f(-t)] \quad \text{Odd Part}$$

# Symmetry property

## Fourier Coefficients of Even Functions

$$\omega = 2\pi/T$$

$$T = 2l$$

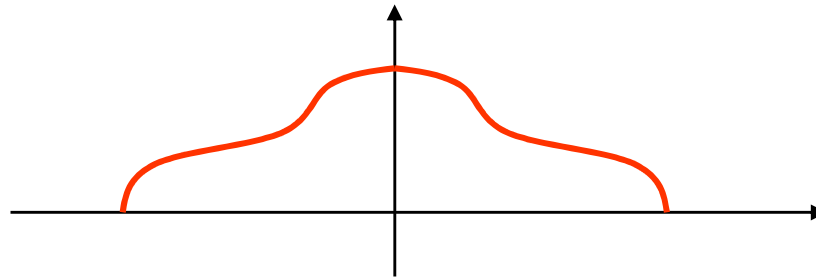
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) dt = \frac{1}{l} \int_{-l}^l f(t) dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega t dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega t dt$$

$$f(t) = f(-t)$$



$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t$$

$$\longrightarrow a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos(n\omega t) dt = \frac{2}{l} \int_0^l f(t) \cos(n\omega t) dt$$

# Symmetry property

## Fourier Coefficients of Odd Functions

$$\omega = 2\pi/T$$

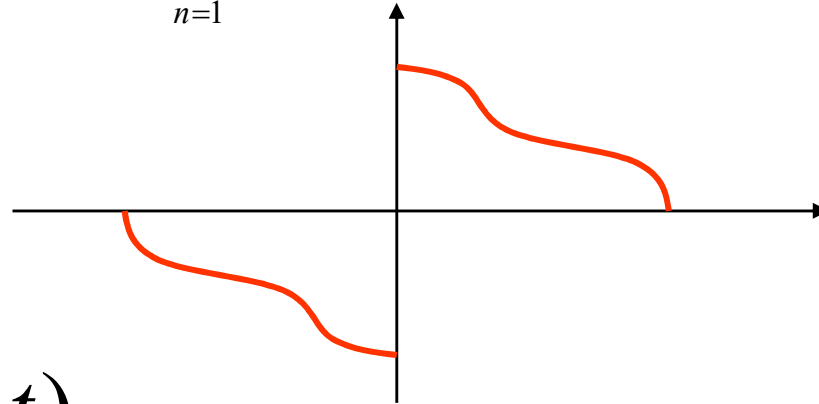
$$T = 2l$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) dt = \frac{1}{l} \int_{-l}^l f(t) dt$$

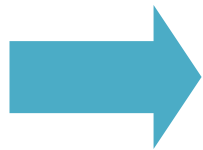
$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega t dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega t dt$$



$$f(t) = -f(-t)$$

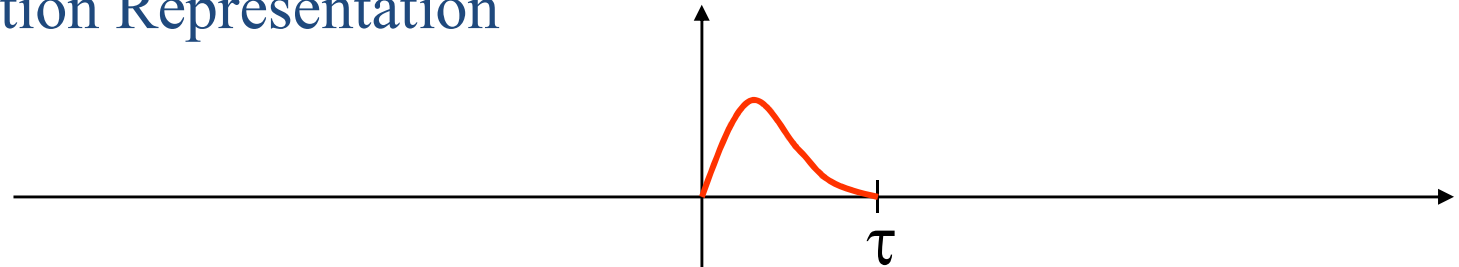
$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$$



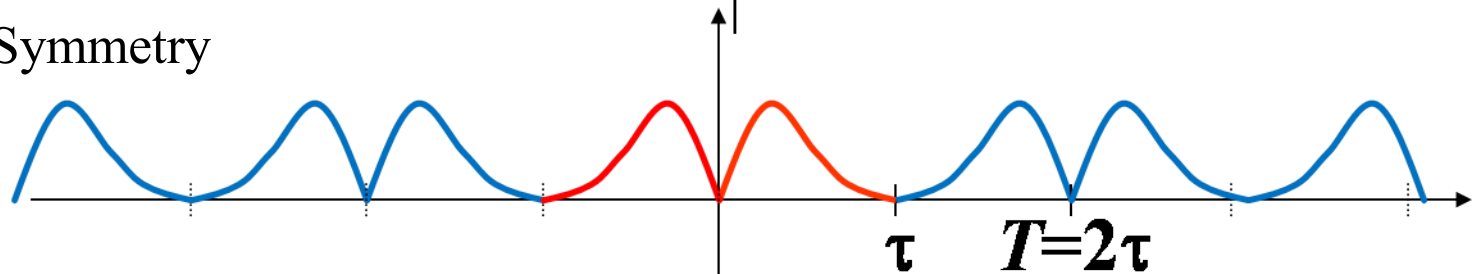
$$b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin(n\omega t) dt = \frac{2}{l} \int_0^l f(t) \sin(n\omega t) dt$$

## Fourier Series - Half-Range Expansions

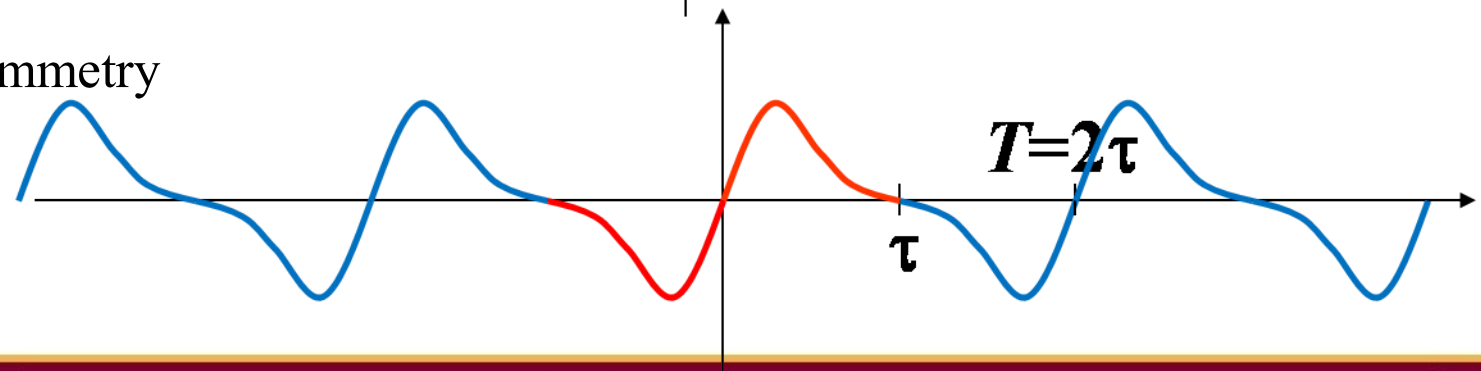
### Non-Periodic Function Representation



### Expansion Into Even Symmetry



### Expansion Into Odd Symmetry





# Fourier series

## DIRICHLET CONDITIONS

Suppose that

1.  $f(x)$  is defined and single valued except possibly at finite number of points in  $(-l,+l)$
2.  $f(x)$  is periodic outside  $(-l,+l)$  with period  $2l=T$
3.  $f(x)$  and  $f'(x)$  are piecewise continuous in  $(-l,+l)$

Then the Fourier series of  $f(x)$  converges to

1.  $f(x)$  if  $x$  is a point of continuity
2.  $[f(x+0)+f(x-0)]/2$  if  $x$  is a point of discontinuity

E.g. For a function defined by  $f(x) = |x|, -\pi < x < \pi$  obtain a Fourier series. Deduce that

$$\omega = 2\pi/T$$

$$T = 2l$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

$$a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) dt = \frac{1}{l} \int_{-l}^l f(t) dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega t dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega t dt$$

Solution  $f(x) = |x|$  is an even function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} |x| dx \qquad f(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} \cos nx$$

$$= \frac{2}{\pi} \int_0^{\pi} x dx = \frac{2}{\pi} \left( \frac{x^2}{2} \right)_0^{\pi} = \pi$$

$x=0$  is a point of continuity,  
by Dirichlet condition the Fourier series

converges to  $f(0)$  and  $f(0)=0$

$$a_n = \frac{2}{\pi} \int_0^{\pi} |x| \cos nxdx = \frac{2}{\pi} \int_0^{\pi} x \cos nxdx$$

$$0 = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left[ \frac{(-1)^n - 1}{n^2} \right]$$

$$= \frac{2}{\pi} \left[ x \left( \frac{\sin nx}{n} \right) - \left( \frac{-\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$0 = \frac{\pi}{2} - \frac{2}{\pi} \left( \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \dots \right)$$

$$= \frac{2}{\pi n^2} [(-1)^n - 1]$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

## HALF RANGE SERIES - COSINE SERIES

A function  $f(x)$  defined in  $(0,l)$  can be expanded as a Fourier series of period containing only cos terms by extending  $f(x)$  suitably in  $(-l,0)$ . (As an even function)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$\omega = 2\pi/T$$

$$T = 2l$$

where

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx, n \geq 0$$

$$a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) dt = \frac{1}{l} \int_{-l}^l f(t) dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega t dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega t dt$$

## HALF RANGE SERIES - SINE SERIES

A function  $f(x)$  defined in  $(0,l)$  can be expanded as a Fourier series of period containing only sine terms by extending  $f(x)$  suitably in  $(-l,0)$ . (As an odd function), period= $T=2l$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\omega = 2\pi/T$$

$$T = 2l$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx, \quad n \geq 1$$

$$a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) dt = \frac{1}{l} \int_{-l}^l f(t) dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega t dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega t dt$$

E.g. Obtain the Fourier expansion of  $(x \sin x)$  as a cosine series in  $(0, \pi)$ . Hence find the value of

$$1 + \frac{2}{1 \cdot 3} - \frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} + \dots$$

### SOLUTION

Given function represents an even function in  $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$\omega = 2\pi/T$$

$$T = 2l$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) dt = \frac{1}{l} \int_{-l}^l f(t) dt$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \sin x dx = \frac{2}{\pi} [x(-\cos x) - 1(-\sin x)]_0^{\pi} = 2$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega t dt$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx dx$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega t dt$$

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx dx \\
 &= \frac{2}{\pi} \int_0^{\pi} x \left[ \frac{1}{2} \sin(1+n)x + \sin(1-n)x \right] dx \\
 &= \frac{1}{\pi} \left[ \left\{ x \left( \frac{-\cos(1+n)x}{1+n} \right) - 1 \cdot \left( \frac{-\sin(1+n)x}{(1+n)^2} \right) \right\}_0^{\pi} \right. \\
 &\quad \left. + \left\{ x \left( \frac{-\cos(1-n)x}{1-n} \right) - 1 \cdot \left( \frac{-\sin(1-n)x}{(1-n)^2} \right) \right\}_0^{\pi} \right] \\
 &= \frac{-1(-1)^{n+1}}{n+1} + \frac{1}{n-1} (-1)^{1-n}
 \end{aligned}$$

Note: even function,  
 $b_n=0$

$$a_n = (-1)^{n-1} \left[ \frac{1}{n-1} - \frac{1}{n+1} \right] = \frac{2(-1)^{n-1}}{n^2-1} \quad \text{If } n \neq 1$$

If  $n = 1$

$$\begin{aligned}
 a_1 &= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x dx = \frac{1}{\pi} \int_0^{\pi} x \sin 2x dx \\
 &= \frac{1}{\pi} \left[ x \left( \frac{-\cos 2x}{2} \right) - 1 \cdot \left( \frac{-\sin 2x}{2^2} \right) \right]_0^{\pi} \\
 &= \frac{1}{\pi} \left[ \frac{-\pi}{2} \right] = \frac{-1}{2}
 \end{aligned}$$

$$x \sin x = 1 - \frac{1}{2} \cos x + 2 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^2 - 1} \cos nx \quad \text{in } (0, \pi)$$

At  $x = \pi/2$ , the above series reduces to

$$1 + 2 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^2 - 1} \cos \left( \frac{n \pi}{2} \right)$$

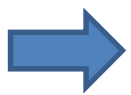
$x = \pi/2$  is a point of continuity

$$f(x) = x \sin x$$

$\therefore$  The given series converges to  $f(\pi/2) = \pi/2$

$$x \sin x = 1 - \frac{1}{2} \cos x + 2 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^2 - 1} \cos nx \quad \text{in } (0, \pi)$$

$$\frac{\pi}{2} = 1 + 2 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^2 - 1} \cos \frac{n\pi}{2}$$



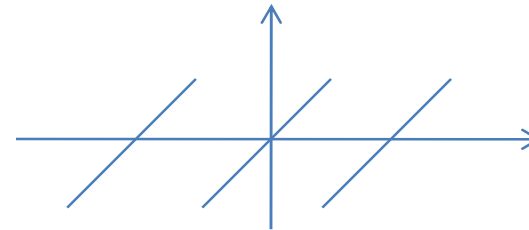
$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots \infty = \frac{\pi - 2}{4}$$



2) Expand  $f(x)=x$ ,  $0 < x < 2$  in half range  
 (a) sine Series (b) Cosine series.

(a) Extend the definition of given function to that of an odd function of period 4

$$f(x) = \begin{cases} x; & -2 < x < 0 \\ x; & 0 < x < 2 \end{cases}$$



$$a_n = 0$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n \pi x}{l} dx$$

$$b_n = \frac{2}{2} \int_0^2 f(x) \sin \frac{n \pi x}{2} dx$$

$$= \left[ x \left( \frac{-\cos \frac{n \pi x}{2}}{\frac{n \pi}{2}} \right) - 1 \left( \frac{-\sin \frac{n \pi x}{2}}{\frac{n^2 \pi^2}{2^2}} \right) \right]_0^2 = \frac{-4(-1)^n}{n \pi}$$

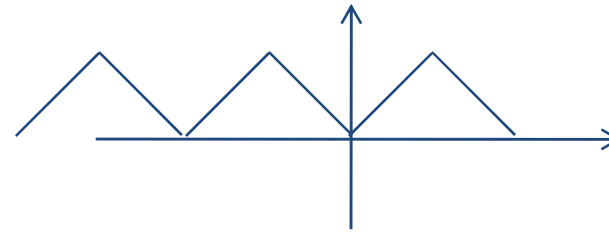
$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{-(-1)^n}{n} \sin \frac{n \pi x}{2}$$

(b) Extend the definition of given function to that of an even function of period 4

$$f(x) = \begin{cases} -x; -2 < x < 0 \\ x; 0 < x < 2 \end{cases}$$

$$b_n = 0$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$



$$a_n = \frac{2}{2} \int_0^2 f(x) \cos \frac{n\pi x}{2} dx$$

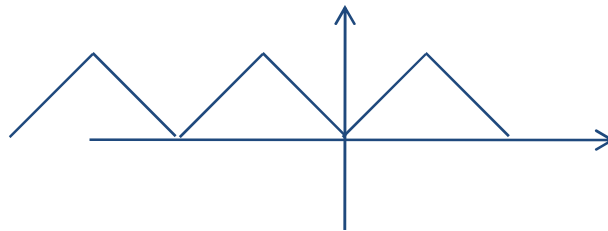
$$= \left[ x \left( \frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right) - 1 \left( \frac{-\cos \frac{n\pi x}{2}}{\frac{n^2 \pi^2}{2^2}} \right) \right]_0^2 = \frac{4[(-1)^n - 1]}{n^2 \pi^2};$$

$$n \neq 0$$

$$f(x) = \begin{cases} -x; & -2 < x < 0 \\ x; & 0 < x < 2 \end{cases}$$

$$a_0 = \int_0^2 x dx = 2$$

$$f(x) = 1 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} \cos \frac{n\pi x}{2}$$



# PARSEVAL'S IDENTITY

$$\int_{-l}^l [f(x)]^2 dx = l \left[ \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

Provided the Fourier series for  $f(x)$  converges uniformly in  $(-l, l)$ .

The Fourier Series for  $f(x)$  in  $(-l, l)$  is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Multiplying both sides of (1) by  $f(x)$  and integrating term from  $-l$  to  $l$   
 (which is justified because  $f(x)$  is uniformly convergent)

$$\int_{-l}^l f(x)^2 dx = \frac{a_0}{2} \int_{-l}^l f(x) dx + \sum_{n=1}^{\infty} a_n \left\{ \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx \right\} + \sum_{n=1}^{\infty} b_n \left\{ \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx \right\}$$

$$a_0 = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) dt = \frac{1}{l} \int_{-l}^l f(t) dt$$

$$= \frac{a_0}{2} \times l a_0 + \sum_{n=1}^{\infty} a_n (l a_n) + \sum_{n=1}^{\infty} b_n (l b_n)$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega t dt$$

$$\therefore \int_{-l}^l f(x)^2 dx = l \left[ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega t dt$$

If  $f(x)$  is defined in  $(0,2l)$  then Parseval's Identity is given by

CASE- a

$$\int_0^{2l} [f(x)]^2 dx = l \left[ \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

CASE- b If half range cosine series in  $(0,l)$  for  $f(x)$  is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n \pi x}{l}$$

Then Parseval's Identity is given by

$$\int_0^l [f(x)]^2 dx = \frac{l}{2} \left[ \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} a_n^2 \right]$$

If the half range Sine series in  $(0, l)$  for  $f(x)$  is

CASE-c

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n \pi x}{l}$$

$$\int_{-l}^l [f(x)]^2 dx = l \left[ \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

Then Parseval's Identity is given by

$$\int_0^l [f(x)]^2 dx = \frac{l}{2} \left[ \sum_{n=1}^{\infty} b_n^2 \right]$$

E.G. Find the Fourier series of periodic function

$$f(x) = x^2 - x \text{ in } (-\pi, \pi)$$

Hence deduce the sum of series

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots \infty$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Assuming that

**SOLUTION**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}. \quad \text{in } (-\pi, \pi)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x^2 - x) \cos nx dx = \frac{1}{\pi} \left[ x^2 \left( \frac{-\sin nx}{n} \right) - 2x \frac{-\cos nx}{n^2} \right]_{-\pi}^{\pi}$$

$$= \frac{4}{n^2} (-1)^n \quad \text{if } n \neq 0.$$

 $\therefore x \cos nx$  is odd function

$$\omega = 2\pi/T.$$

$$T = 2l$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x^2 - x) \sin nx dx = 0 - \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx$$

$$= -\frac{1}{\pi} \left[ x \left( \frac{-\cos nx}{n} \right) - 1 \left( \frac{-\sin nx}{n^2} \right) \right]_{-\pi}^{\pi} = \frac{2}{n} (-1)^n$$

 $\therefore x^2 \sin nx$  is odd function

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx. \quad \text{in } (-\pi, \pi)$$

Using the Parseval's Identity

$$p^2 = \left[ \frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right].$$

$$p^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (x^2 - x)^2 dx = \frac{\pi^4}{5} + \frac{\pi^2}{3}$$

$$a_0 = \frac{2\pi^2}{3}; a_n = \frac{4}{n^2} (-1)^n; b_n = \frac{2}{n} (-1)^n$$

$$\therefore \frac{\pi^4}{5} + \frac{\pi^2}{3} = \frac{1}{4} \frac{4\pi^4}{9} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{16}{n^4} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \quad \therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$



E.g. 2 By using sine series for  $f(x)=1$  in  $0 < x < \pi$ .

Show that 
$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \infty$$

SOLUTION

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n \pi x}{l} = \sum_{n=1}^{\infty} b_n \sin (nx)$$

$$\omega = 2\pi/T$$

$$T = 2l$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} \sin nx dx = \frac{2}{\pi} \left\{ \frac{-\cos nx}{n} \right\}_0^{\pi}$$

$$= \frac{2}{n\pi} [1 - (-1)^n] \quad \text{for } n \neq 0.$$

$$f(x) = 1 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin (nx)$$

By Parseval's Identity

$$p^2 = \left[ \frac{1}{2} \sum_{n=1}^{\infty} b_n^2 \right].$$

$$p^2 = \frac{1}{\pi} \int_0^{\pi} 1^2 dx = 1 = \frac{1}{2} \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} [1 - (-1)^n]^2$$

$$\rightarrow \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \infty$$