SSCE1693 ENGINEERING MATHEMATICS

CHAPTER 7:

MATRIX ALGEBRA

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7.0 Matrix Algebra

Definition 7.1: Matrix

Matrix is a rectangular array of numbers which called elements

7.1 Elementary Row Operations (ERO)

- Important method to find the inverse of a matrix and to solve the system of linear equations.

- The following notations will be used while applying ERO

1. Interchange the i^{th} row with the j^{th} row of the matrix. This process is denoted as $\boldsymbol{B_i} \leftrightarrow \boldsymbol{B_j}$.
2. Multiply the i^{th} row of the matrix with the scalar k where $k \neq 0$. This process is denoted as $\boldsymbol{kB_i}$.
3. Add the i^{th} row, that is multiplied by the scalar h to the j^{th} row that has been multiplied by the scalar k, where $h \neq 0,$ and $k \neq 0$. This process can be denoted as $\boldsymbol{hB_i} + \boldsymbol{kB_j}$. The purpose of this process is to change the elements in the i^{th} row.

Example 7.1:

Given the matrix $A = \begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & 1 \\ -3 & 1 & 2 \end{pmatrix}$, perform the following operations consecutively: $B_1 \leftrightarrow B_2$, $B_2 + (-2)B_1$, $B_3 + 3B_1$, $B_3 + (-7)B_2$ and $-\frac{1}{2}B_3$.

Solution:

$$\begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & 1 \\ -3 & 1 & 2 \end{pmatrix} \xrightarrow[B_1 \leftrightarrow B_2]{} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ -3 & 1 & 2 \end{pmatrix} \xrightarrow[B_2 + (-2)B_1]{} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -3 & 1 & 2 \end{pmatrix} \xrightarrow[B_3 + 3B_1]{}$$

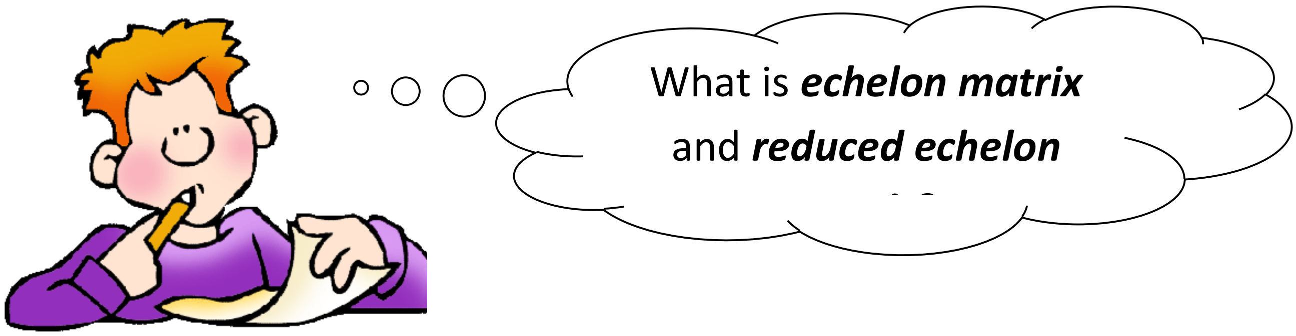
$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 7 & 5 \end{pmatrix} \xrightarrow[B_3 + (-7)B_2]{} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow[-\frac{1}{2}B_3]{} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Notes:

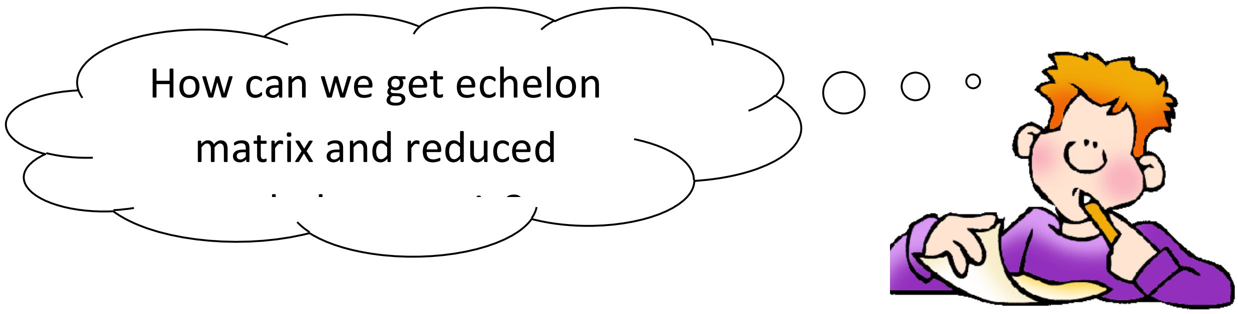
If the matrix A is transformed to the matrix B by using ERO, then the matrix A is called *equivalent matrix* to

Definition 7.2: Rank of a Matrix

The rank of a matrix is the number of row that is non zero in that *echelon matrix* or *reduced echelon matrix*. The rank of matrix A is denoted as $p(A)$.



$\begin{pmatrix} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow p(A) = 3$	$\begin{pmatrix} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \end{pmatrix} \Rightarrow p(A) = 3$
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow p(A) = 3$	$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow p(A) = 3$
$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow p(A) = 2$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow p(A) = 2$
Example of Echelon Matrix and its rank of matrix	Example of Reduced Echelon Matrix and its rank of matrix



Example 7.2:

Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{pmatrix}$ obtain

a) Echelon matrix
b) Reduced echelon matrix
c) Rank of matrix A

Solution:

a) $$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{pmatrix} \xrightarrow[B_3+(-3)B_1]{B_2+(-2)B_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 0 & -5 & -10 \end{pmatrix} \xrightarrow[\left(-\frac{1}{7}\right)B_2 \;\; \left(-\frac{1}{5}\right)B_3]{}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4/7 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow[B_3+(-1)B_2]{} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4/7 \\ 0 & 0 & 10/7 \end{pmatrix} \xrightarrow[7/10 B_3]{} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4/7 \\ 0 & 0 & 1 \end{pmatrix}$$

b) $$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4/7 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[B_2+\left(-\frac{4}{7}\right)B_3]{B_1+(-2)B_2} \begin{pmatrix} 1 & 0 & 13/7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[B_1+\left(-\frac{13}{7}\right)B_3]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c) $p(A) = 3$

7.2 Determinant of a Matrix

7.2.1 Determinant

- A scalar value that can be used to find the inverse of a matrix.
- The inverse of the matrix will be used to solve a system of linear equations.

Definition 7.3 : Determinant

The determinant of a matrix A is a scalar value and denoted by $|\boldsymbol{A}|$ or $\mathbf{det}\,(\boldsymbol{A})$.

1. The determinant of a 2x2 matrix is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

2. The determinant of a 3x3 matrix is defined by

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

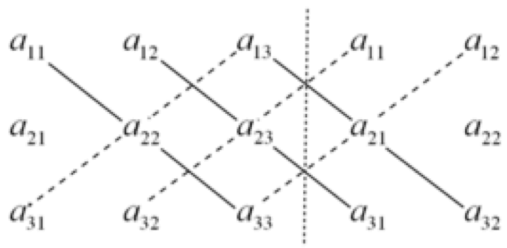


Figure 7.1: The determinant of a 3x3 matrix can be calculated by its diagonal

3. The determinant of a n x n matrix can be calculated by using **cofactor expansion**. (Note: *This involves minor and cofactor so we will see this method after reviewing minor and cofactor of a matrix)*

Definition 7.4: Minor

If

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \ddots & a_{ij} & \ddots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

then the **minor** of $\boldsymbol{a_{ij}}$, denoted by $\mathbf{D_{ij}}$ is the determinant of the submatrix that results from removing the i^{th} row and j^{th} column of **A**.

Example 7.3:

Find the minor $\mathbf{D_{12}}$ for matrix **A**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Solution:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \mathbf{D}_{\mathbf{12}} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$$

Example 7.4:

Given $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \\ 2 & 4 & -5 \end{pmatrix}$. Calculate the minor of a_{11} and a_{32}.

Solution:

$$D_{11} = \begin{vmatrix} -1 & 3 \\ 4 & -5 \end{vmatrix} = (-1)(-5) - (4)(3) = -7$$

$$D_{32} = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = (1)(3) - (0)(2) = 3$$

7.2.3 Cofactor

Definition 7.5: Cofactor

Example 7.5:

Find the cofactor A_{23} from the given matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 3 & 0 & 5 \\ -1 & 9 & 11 \end{bmatrix}$$

Solution:

$$A_{23} = (-1)^{2+3} D_{23}$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 4 \\ -1 & 9 \end{vmatrix} = (-1)\big(9 - (-4)\big) = -13$$

Example 7.6:

From Example 7.4, find the cofactor of a_{11} and a_{32}

Solution:

$$A_{11} = (-1)^{1+1} D_{11} = (-1)^2 \begin{vmatrix} -1 & 3 \\ 4 & -5 \end{vmatrix} = (1)(-7) = -7$$

$$A_{32} = (-1)^{3+2} D_{32} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = (-1)(3) = -3$$

7.2.4 Cofactor Expansion

Theorem 7.1: Cofactor Expansion

If A is an n x n matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The determinant of A (det(A)) can be written as the sum of its cofactors multiplied by the entries that generated them.

a) Cofactor expansion along the j^{th} column

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} = \sum_{i=1}^{n} a_{ij}A_{ij}$$

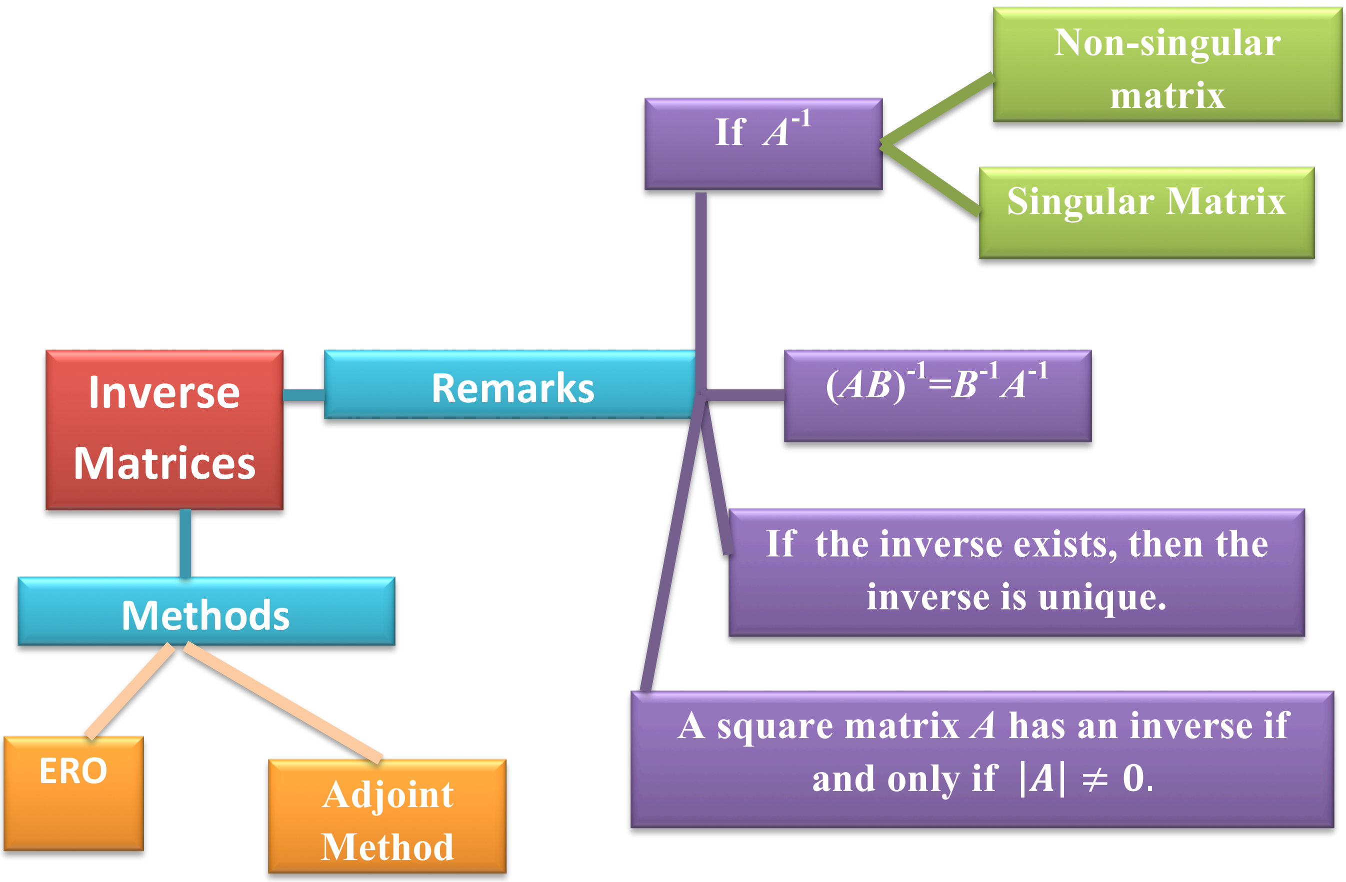
b) Cofactor expansion along the i^{th} row

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

7.3 Inverse Matrices

Definition 7.6: Inverse Matrix

If A and B are $n \times n$ matrices, then the matrix B is the



7.3.1 Finding Inverse Matrices using ERO

STEP 1:

Write AI in the form of augmented matrix $(A|I)$.

STEP 2:

Example 7.7:

Calculate the inverse of the following matrix

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 3 & 5 & 1 \\ 6 & 4 & 2 \end{pmatrix}$$

Solution:

STEP 1:

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 3 & 5 & 1 & 0 & 1 & 0 \\ 6 & 4 & 2 & 0 & 0 & 1 \end{array}\right)$$

STEP 2:

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 3 & 5 & 1 & 0 & 1 & 0 \\ 6 & 4 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[B_3+(-6)B_1]{B_2+(-3)B_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 11 & -8 & -3 & 1 & 0 \\ 0 & 16 & -16 & -6 & 0 & 1 \end{array}\right) \xrightarrow[\substack{B_2/11 \\ B_3/16}]{}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -8/11 & -3/11 & 1/11 & 0 \\ 0 & 1 & -1 & -3/8 & 0 & 1/16 \end{array}\right) \xrightarrow[B_3+(-1)B_2]{B_1+(2)B_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 17/11 & 5/11 & 2/11 & 0 \\ 0 & 1 & -8/11 & -3/11 & 1/11 & 0 \\ 0 & 0 & -3/11 & -9/88 & -1/11 & 1/16 \end{array}\right)$$

$$\xrightarrow[-11B_3/3]{} \left(\begin{array}{ccc|ccc} 1 & 0 & 17/11 & 5/11 & 2/11 & 0 \\ 0 & 1 & -8/11 & -3/11 & 1/11 & 0 \\ 0 & 0 & 1 & 3/8 & 1/3 & -11/48 \end{array}\right) \xrightarrow[B_2+(8/11)B_3]{B_1+(-17/11)B_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/8 & -1/3 & 17/48 \\ 0 & 1 & 0 & 0 & 1/3 & -1/6 \\ 0 & 0 & 1 & 3/8 & 1/3 & -11/48 \end{array}\right)$$

STEP 3:

$$A^{-1} = \begin{pmatrix} -{}^{1}/_{8} & -{}^{1}/_{3} & {}^{11}/_{48} \\ 0 & {}^{1}/_{3} & -{}^{1}/_{6} \\ {}^{3}/_{8} & {}^{1}/_{3} & -{}^{11}/_{48} \end{pmatrix} = \frac{1}{48}\begin{pmatrix} -6 & -16 & 17 \\ 0 & 16 & -8 \\ 18 & 16 & -11 \end{pmatrix}$$

7.3.2 Finding Inverse Matrices using Adjoint

Method

Definition 7.7: Adjoint of a Matrix

The **adjoints of a square matrix *A*** is the transpose of cofactor matrix which can be obtained by interchanging every element a_{ij} with the cofactor c_{ij} and denoted as

$$adj(\boldsymbol{A}) = [c_{ij}]^T = [c_{ij}].$$

If $|A| \neq 0$, then A^{-1} exists. Therefore the inverse matrix is,

$$A^{-1} = \frac{1}{|A|}\text{adj}(A).$$

STEPS TO FIND THE INVERSE MATRIX USING ADJOINT METHOD.

STEP 1: Calculate the determinant of A.

i) If $|A| = 0$, stop the calculation because the inverse does not exist.
ii) If $|A| \neq 0$, continue to STEP 2.

STEP 2: Calculate the cofactor matrix $[c_{ij}]$.

STEP 3: Find the adjoint matrix A by finding the transpose of the cofactor matrix $[c_{ij}]$, that is

$$\text{adj}(A) = [c_{ij}]^T = [c_{ij}].$$

STEP 4: Substitute the results from STEP 1 to STEP 3 in the formula

$$A^{-1} = \frac{1}{|A|}\text{adj}(A).$$

Example 7.12:

Calculate the inverse of the following matrix

$$A = \begin{bmatrix} 4 & 2 & 1 \\ -2 & -6 & 3 \\ -7 & 5 & 0 \end{bmatrix}$$

Solution:

Step 1: Calculate the determinant of A.

$$|A| = -154 \neq 0$$

Step 2*:* Find the cofactor matrix.

$$C_{11} = (-1)^2 \begin{vmatrix} -6 & 3 \\ 5 & 0 \end{vmatrix} = -15 \qquad C_{12} = (-1)^3 \begin{vmatrix} -2 & 3 \\ -7 & 0 \end{vmatrix} = -21 \qquad C_{13} = (-1)^4 \begin{vmatrix} -2 & -6 \\ -7 & 5 \end{vmatrix} = -52$$

$$C_{21} = (-1)^3 \begin{vmatrix} 2 & 1 \\ 5 & 0 \end{vmatrix} = 5 \qquad C_{22} = (-1)^4 \begin{vmatrix} 4 & 1 \\ -7 & 0 \end{vmatrix} = 7 \qquad C_{23} = (-1)^5 \begin{vmatrix} 4 & 2 \\ -7 & 5 \end{vmatrix} = -34$$

$$C_{31} = (-1)^4 \begin{vmatrix} 2 & 1 \\ -6 & 3 \end{vmatrix} = 12 \qquad C_{32} = (-1)^5 \begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} = -14 \qquad C_{33} = (-1)^6 \begin{vmatrix} 4 & 2 \\ -2 & -6 \end{vmatrix} = -20$$

$$\therefore \text{Matrix of cofactor, } C = \begin{pmatrix} -15 & -21 & -52 \\ 5 & 7 & -34 \\ 12 & -14 & -20 \end{pmatrix}$$

Step 3: Adjoint of A

$$Adj(A) = \begin{pmatrix} -15 & -21 & -52 \\ 5 & 7 & -34 \\ 12 & -14 & -20 \end{pmatrix}^T = \begin{pmatrix} -15 & 5 & 12 \\ -21 & 7 & -14 \\ -52 & -34 & -20 \end{pmatrix}$$

Step 4: Find A^{-1}

$$A^{-1} = -\frac{1}{154}\begin{pmatrix} -15 & 5 & 12 \\ -21 & 7 & -14 \\ -52 & -34 & -20 \end{pmatrix}$$

<u>EXERCISE 7.1:</u>

1. Calculate the inverse of the following matrices by using
   (i) Elementary Row Operations (ERO) methods
   (ii) Adjoint Method

a) $\begin{pmatrix} -3 & -1 & 6 \\ 2 & 1 & -4 \\ -5 & -2 & 11 \end{pmatrix}$

b) $\begin{pmatrix} -3 & 1 & 2 \\ 2 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & -4 \\ -5 & 2 & 1 \end{pmatrix}$

d)

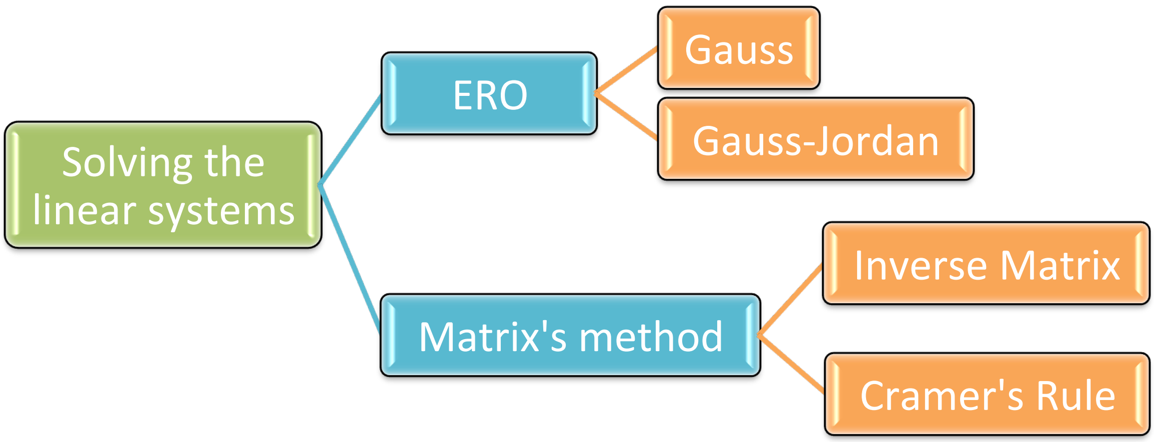
7.4 SYSTEMS OF LINEAR EQUATIONS

A system of linear equations with m linear equations and n number of variables can be written as,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

A solution to a linear system are real values of $x_1, x_2, x_3, \ldots, x_n$ which satisfy every equations in the linear systems.

If the solution does not exist, then the system is inconsistent.



- **Non-homogeneous system**
  - *p(A)=p(A|b)* = number of variables ↦ the system has a unique solution.
  - *p(A)* = *p(A|b)* < number of variables ↦ the system has many solutions.
  - *p(A)* < *p(A|b)* = the number of variables ↦ the system has no solution.
- **Homogeneous system**
  - *p(A)* = number of variables ↦ the system has trivial solution.
  - *p(A)* < the number of variables ↦ the system has many solutions.

7.4.1 Gauss Elimination Method

Gauss Elimination is a method of solving a linear

Example 7.13:

Solve the following system by using Gauss Elimination method.

$$2x_1 - 3x_2 - x_3 + 2x_4 + 3x_5 = 4$$

$$4x_1 - 4x_2 - x_3 + 4x_4 + 11x_5 = 4$$

$$2x_1 - 5x_2 - 2x_3 + 2x_4 - x_5 = 9$$

$$2x_2 + x_3 + 4x_5 = -5$$

Solution:

STEP 1: Construct the augmented matrix

$$\left(\begin{array}{ccccc|c} 2 & -3 & -1 & 2 & 3 & 4 \\ 4 & -4 & -1 & 4 & 11 & 4 \\ 2 & -5 & -2 & 2 & -1 & 9 \\ 0 & 2 & 1 & 0 & 4 & -5 \end{array}\right)$$

STEP 2: Use ERO to transform this matrix into the following echelon matrix

$$\left(\begin{array}{ccccc|c} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 0 & 1 & 1/2 & 0 & 5/2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

STEP 3: Solve using back substitution

$$x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 + x_4 + \frac{3}{2}x_5 = 2$$

$$x_2 + \frac{1}{2}x_3 + \frac{5}{2}x_5 = -2$$

$$x_5 = 1$$

Set $x_3 = s$ and $x_4 = t$,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -(25/4) - (1/4)s - t \\ -(9/2) - (1/2)s \\ s \\ t \\ 1 \end{pmatrix}$$

7.4.2 Gauss-Jordan Elimination Method

Gauss Elimination is a method of solving a linear system $A\mathbf{x} = \mathbf{b}$ by bringing the augmented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{21} & \cdots & a_{1n} & b_1 \end{array}\right.$$

Example 7.14:

By using the same matrix in Example 7.13, find the solution for the linear system by using Gauss-Jordan Elimination method.

Solution:

From STEP 2 in Example 7.13, we can use ERO to find the reduced echelon matrix for the augmented matrix.

$$\left(\begin{array}{ccccc|c} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 0 & 1 & 1/2 & 0 & 5/2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[B_2+\left(-\frac{5}{2}\right)B_3]{B_1+\left(\frac{3}{2}\right)B_2} \left(\begin{array}{ccccc|c} 1 & 0 & 1/4 & 1 & 21/4 & -1 \\ 0 & 1 & 1/2 & 0 & 0 & -9/2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[B_1+\left(-\frac{21}{4}\right)B_3]{} \left(\begin{array}{ccccc|c} 1 & 0 & 1/4 & 1 & 0 & -25/4 \\ 0 & 1 & 1/2 & 0 & 0 & -9/2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

From the reduced echelon matrix, we will get the following equations

$x_5 = 1$

$x_2 = -(9/2) - (1/2)x_3$

$x_1 = -(25/4) - (1/4)x_3 - x_4$

By setting $x_3 = s$ and $x_4 = t$,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -(25/4) - (1/4)s - t \\ -(9/2) - (1/2)s \\ s \\ t \\ 1 \end{pmatrix}$$

EXERCISE 7.2:

1. Solve the linear system by using
   (i) Gauss elimination method
   (ii) Gauss-Jordan elimination method

a) $y + z = 2,$
$2x + 3z = 5,$
$x + y + z = 3$

b) $x - 2y + 3z = -2,$
$-x + y - 2z = 3,$
$2x - y + 3z = 1$

7.4.3 Inverse Matrix Method

If $|A| \neq 0$ and $A\mathbf{x} = \mathbf{b}$ represents the linear equations where A is an $n{\times}n$ matrix and B is an $n{\times}1$ matrix, then the solution for the system is given as

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Example 7.15:

Use the method of inverse matrix to determine the solution to the following system of linear equations.

$$3x_1 - x_2 + 5x_3 = -2$$

$$-4x_1 + x_2 + 7x_3 = 10$$

$$2x_1 + 4x_2 - x_3 = 3$$

Solution:

STEP 1: Check whether $|A| \neq 0$.

$$\underbrace{\begin{bmatrix} 3 & -1 & 5 \\ -4 & 1 & 7 \\ 2 & 4 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -2 \\ 10 \\ 3 \end{bmatrix}}_{\mathbf{b}}$$

$$\begin{aligned} |A| &= (3)(1)(-1) + (-1)(7)(2) + (5)(-4)(4) \\ &\quad -(-1)(-4)(-1) - (2)(1)(5) - (4)(7)(3) \\ &= -187 \neq 0 \end{aligned}$$

STEP 2: Find A^{-1}.by using Adjoint Method or ERO.

i) Matrix of cofactor and adj(A),

$$C = \begin{pmatrix} \begin{vmatrix} 1 & 7 \\ 4 & -1 \end{vmatrix} & -\begin{vmatrix} -4 & 7 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} -4 & 1 \\ 2 & 4 \end{vmatrix} \\ -\begin{vmatrix} -1 & 5 \\ 4 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 5 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix} \\ \begin{vmatrix} -1 & 5 \\ 1 & 7 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ -4 & 7 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ -4 & 1 \end{vmatrix} \end{pmatrix}$$

$$C = \begin{pmatrix} -29 & 10 & -18 \\ 19 & -13 & -14 \\ -12 & -41 & -1 \end{pmatrix}, \text{adj}(A) = C^T = \begin{pmatrix} -29 & 19 & -12 \\ 10 & -13 & -41 \\ -18 & -14 & -1 \end{pmatrix}$$

ii) $A^{-1} = \frac{1}{-187}\begin{pmatrix} -29 & 19 & -12 \\ 10 & -13 & -41 \\ -18 & -14 & -1 \end{pmatrix}$

$$= \begin{pmatrix} \frac{29}{187} & -\frac{19}{187} & \frac{12}{187} \\ -\frac{10}{187} & \frac{13}{187} & \frac{41}{187} \\ \frac{18}{187} & \frac{14}{187} & \frac{1}{187} \end{pmatrix}$$

STEP 3: Solution for **x** is given by

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{28}{187} & -\frac{19}{187} & \frac{12}{187} \\ -\frac{10}{187} & \frac{13}{187} & \frac{41}{187} \\ \frac{18}{187} & \frac{14}{187} & \frac{1}{187} \end{bmatrix}\begin{bmatrix} -2 \\ 10 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{212}{187} \\ \frac{273}{187} \\ \frac{107}{187} \end{bmatrix}$$

EXERCISE 7.3:

1) Solve the following system linear equations by using Inverse Matrix Method

(a) $x_1 + x_2 + 2x_3 = 7$

$x_1 - x_2 - 3x_3 = -6$

$2x_1 + 3x_2 + x_3 = 4$

(b) $2x_1 + 3x_2 + x_3 = 11$

$2x_1 - 2x_2 - 3x_3 = 5$

$3x_1 - 5x_2 + 2x_3 = -3$

7.4.3 Cramer's Rule

Given the system of linear equations $A\mathbf{x} = \mathbf{b}$, where A is an n x n matrix, $\mathbf{x}$ and $\mathbf{b}$ are $n\times 1$ matrices. If $|A| \neq 0$, then the solution to the system is given by,

$$x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, \dots, x_n = \frac{|A_n|}{|A|}$$

for $i = 1,2,\dots,n$ where A_i is the matrix found by replacing the i^{th} column of A with $\mathbf{b}$.

Example 7.16:

Use Cramer's rule to determine the solution to the following system of linear equations.

$$3x_1 - x_2 + 5x_3 = -2$$

$$-4x_1 + x_2 + 7x_3 = 10$$

$$2x_1 + 4x_2 - x_3 = 3$$

Solution:

1. Test whether $|A| \neq 0$, or not.

$$\underbrace{\begin{bmatrix} 3 & -1 & 5 \\ -4 & 1 & 7 \\ 2 & 4 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -2 \\ 10 \\ 3 \end{bmatrix}}_{\mathbf{b}}$$

$$\begin{aligned} |A| &= (3)(1)(-1) + (-1)(7)(2) + (5)(-4)(4) \\ &\quad -(-1)(-4)(-1) - (2)(1)(5) - (4)(7)(3) \\ &= -187 \neq 0 \end{aligned}$$

By using the Cramer's rule,

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} \boxed{-2} & -1 & 5 \\ \boxed{10} & 1 & 7 \\ \boxed{3} & 4 & -1 \end{vmatrix}}{-187} = -\frac{212}{187}$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 3 & \boxed{-2} & 5 \\ -4 & \boxed{10} & 7 \\ 2 & \boxed{3} & -1 \end{vmatrix}}{-187} = \frac{273}{187}$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 3 & -1 & \boxed{-2} \\ -4 & 1 & \boxed{10} \\ 2 & 4 & \boxed{3} \end{vmatrix}}{-187} = \frac{107}{187}$$

EXERCISE 7.4:

Solve the following system linear equations by using Cramer's Rule Method.

(a)
$$x_1 + x_2 + 2x_3 = 7$$
$$x_1 - x_2 - 3x_3 = -6$$
$$2x_1 + 3x_2 + x_3 = 4$$

(b)
$$2x_1 + 3x_2 + x_3 = 11$$
$$2x_1 - 2x_2 - 3x_3 = 5$$
$$3x_1 - 5x_2 + 2x_3 = -3$$

7.5 EIGENVALUES & EIGENVECTORS

7.5.1 Eigenvalues & Eigenvectors

Definition 7.8: Eigenvalues & Eigenvectors

Let A be an n x n matrix and the scalar λ is called an eigenvalue of A if there is a non zero vector $\boldsymbol{x}$ such that

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

Example 7.17:

Show that $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$. Hence, find the corresponding eigenvalue.

Solution:

$$A\boldsymbol{x} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\boldsymbol{x}.$$

Therefore, the corresponding eigenvalue is 3.

Definition 7.9: Eigenvalues

The eigenvalues of an n x n matrix A are the n zeroes of the polynomial $P(\lambda) = |A - \lambda I|$ or equivalently the n roots of the

Example 7.18:

Determine the eigenvalues and eigenvector for the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}.$$

Solution:

Step 1: Write down the characteristic equation.

$$\left| \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix} \right| = 0$$

$$P(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Step 2: Find the roots/eigenvalues

By using trial and error, we can take $\lambda = 1$ and it will give

$$P(1) = (1)^3 - 6(1)^2 + 11(1) - 6 = 0$$

Thus $(\lambda - 1)$ is a factor for $P(\lambda)$.

By using long division, the other two factors are $(\lambda - 2)$ and $(\lambda - 3)$. Therefore,

$$P(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

Hence, the eigenvalues of matrix A are $\lambda = 1, 2, 3$.

Step 3: Use the eigenvalues to find the eigenvectors using formula $A\boldsymbol{x} = \lambda\boldsymbol{x}$**.**

When $\lambda = 1$:

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (1)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Using ERO

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \xrightarrow[B_1+(-1)B_3]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \xrightarrow[B_3+(1)B_1]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow[B_3+(-1)B_2]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} x_1 &= 0 \\ x_2 &= -2x_3 = -2k \\ x_3 &= k \end{aligned}$$

Therefore,

$\boldsymbol{x} = \begin{pmatrix} 0 \\ -2k \\ k \end{pmatrix} = k\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ and the corresponding eigenvector is $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

<u>When $\lambda = 2$:</u>

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (2)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Using ERO

$$\begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix} \xrightarrow[B_3+(-1)B_1]{} \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \quad \begin{matrix} 2x_3 = -x_3 = 0 \ \Rightarrow \ x_3 = 0 \\ x_2 = k \\ -x_1 + x_2 + 2x_3 = 0 \Rightarrow x_1 = x_2 = k \end{matrix}$$

Therefore

$$\boldsymbol{x} = \begin{pmatrix} k \\ k \\ 0 \end{pmatrix} = k\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$ and the corresponding eigenvector is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

<u>When $\lambda = 3$:</u>

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Using ERO

$$\begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \xrightarrow[B_3 + (-\frac{1}{2})B_1]{} \begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & \frac{1}{2} & -1 \end{pmatrix} \xrightarrow[B_3 + (\frac{1}{2})B_2]{} \begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \quad \begin{matrix} x_3 = \mathrm{k} \\ -x_2 + 2x_3 = 0 \ \Rightarrow \ x_2 = 2k \\ -2x_1 + x_2 + 2x_3 = 0 \ \Rightarrow \ x_1 = 2k \end{matrix}$$

Therefore

$$\boldsymbol{x} = \begin{pmatrix} 2k \\ 2k \\ k \end{pmatrix}$$