SSCE1693 ENGINEERING MATHEMATICS

CHAPTER 2:
DIFFERENTIATION

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2.1 Differentiation of Hyperbolic Functions

2.2 Differentiation of Inverse Trigonometric Functions

2.3 Differentiation of Inverse Hyperbolic Functions

Revision: Methods of differentiation

1. Chain rule
2. Product differentiation
3. Quotient differentiation
4. Implicit differentiation

2.1 Differentiation of Hyperbolic Functions

***Recall:* Definition:**

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$$

$$\text{cosech}\, x = \frac{1}{\sinh x}$$

$$\text{sech}\, x = \frac{1}{\cosh x}$$

Derivatives of hyperbolic functions

Example 2.1: Find the derivatives of

(a) sinh *x* (b) cosh *x* (c) tanh *x*

Solution:

(a) $$\frac{d}{dx}\sinh x = \frac{d}{dx}\left(\frac{e^{x} - e^{-x}}{2}\right)$$

$$= \frac{1}{2}(e^{x} + e^{-x}) = \cosh x$$

(b) $$\frac{d}{dx}\cosh x = \frac{d}{dx}\left(\frac{e^{x} + e^{-x}}{2}\right)$$

$$= \frac{1}{2}(e^{x} - e^{-x}) = \sinh x$$

(c) $\frac{d}{dx}\tanh x = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

Using quotient diff:

$$= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{(e^x + e^{-x})^2}$$

$$= \frac{4}{(e^x + e^{-x})^2} = \left(\frac{2}{(e^x + e^{-x})}\right)^2$$

$$= \left(\frac{1}{\cosh x}\right)^2 = \operatorname{sech}^2 x$$

Using the same methods, we can obtain the derivatives of the other hyperbolic functions and these gives us the standard derivatives.

Standard Derivatives

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$
$\tanh x$	$\text{sech}^2 x$
$\text{sech}\, x$	$-\text{sech}\, x \tanh x$
$\text{cosech}\, x$	$-\text{cosech}\, x \coth x$
$\coth x$	$-\text{cosech}^2 x$

Example 2.2:

1. Find the derivatives of the following functions:

   a) $y = \dfrac{x^3}{\sinh x}$

   b) $y = \tanh^3 2x$

   c) $y = e^{\cosh 4x^2}$

2. Find the derivatives of the following functions:

   (a) $y = \cosh(3x)$ (b) $r = \sinh(2t^2 - 1)$

   (c) $g(x) = (x-1)^3 \operatorname{sech}^2 x$ (d) $y = \tanh(\ln x)$

3. (Implicit differentiation)

   Find $\dfrac{dy}{dx}$ from the following expressions:

   (a) $x = y^2 \sinh 4x + \cosh y$

   (b) $y = \tanh(x + y)$

2.2 Differentiation Involving Inverse Trigonometric Functions

***Recall*: Definition of inverse trigonometric functions**

Function	Domain	Range
$\sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$\cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$\tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$\sec^{-1} x$	$\lvert x \rvert \ge 1$	$0 \le y < \frac{\pi}{2} \cup \frac{\pi}{2} < y < \pi$
$\cot^{-1} x$	$-\infty \le x \le \infty$	$0 < y < \pi$
$\operatorname{cosec}^{-1} x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} < y < 0 \cup 0 < y < \frac{\pi}{2}$

Derivatives of Inverse Trigonometric Functions

Standard Derivatives:

1. $$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

2. $$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

3. $$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

4. $$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

5. $$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

6. $$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

Derivatives of $y = \sin^{-1} x$. *(proof)*

Recall: $y = \sin^{-1} x \Leftrightarrow x = \sin y$ for $x \in [-1, 1]$ and $y \in [-\pi/2, \pi/2]$.

Because the sine function is differentiable on $[-\pi/2, \pi/2]$, the inverse function is also differentiable.

To find its derivative we proceed implicitly:

Given $\sin y = x$. Differentiating w.r.t. x:

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

$$\cos y \frac{dy}{dx} = 1$$

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{\cos y}$$

Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $\cos y \ge 0$, so

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

Example 2.3:

1. Differentiate each of the following functions.

   (a) $f(x) = \tan^{-1}\sqrt{x}$

   (b) $g(t) = \sin^{-1}(1-t)$

   (c) $h(x) = \sec^{-1} e^{2x}$

2. Find the derivative of:

   (a) $y = (\tan^{-1} x^2)^4$

   (b) $f(x) = \ln(\sin^{-1} 4x)$

3. Find the derivative of $y = \tan^{-1}(\tan(3t^2 - 1))$.

4. Find the derivative $\dfrac{dy}{dx}$ if

   (a) $x\tan^{-1} y = x^2 + y$

   (b) $\sin^{-1}(xy) + \dfrac{\pi}{2} = \cos^{-1} y$

Summary

If u is a differentiable function of x, then

1. $$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \bullet \frac{du}{dx}$$

2. $$\frac{d}{dx}(\cos^{-1} u) = -\frac{1}{\sqrt{1-u^2}} \bullet \frac{du}{dx}$$

3. $$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2} \bullet \frac{du}{dx}$$

4. $$\frac{d}{dx}(\cot^{-1} u) = -\frac{1}{1+u^2} \bullet \frac{du}{dx}$$

5. $$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}} \bullet \frac{du}{dx}$$

6. $$\frac{d}{dx}(\csc^{-1} u) = -\frac{1}{|u|\sqrt{u^2-1}} \bullet \frac{du}{dx}$$

2.3 Derivatives of Inverse Hyperbolic Functions

Recall: Inverse Hyperbolic Functions

Function	*Domain*	*Range*
$y = \sinh^{-1} x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$y = \cosh^{-1} x$	$[1, \infty)$	$[0, \infty)$
$y = \tanh^{-1} x$	$(-1, 1)$	$(-\infty, \infty)$
$y = \coth^{-1} x$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \text{sech}^{-1} x$	$(0, 1]$	$[0, \infty)$
$y = \text{cosech}^{-1} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Function	**Logarithmic form**
$y = \sinh^{-1} x$	$\ln\left(x + \sqrt{x^2 + 1}\right)$
$y = \cosh^{-1} x$	$\ln\left(x + \sqrt{x^2 - 1}\right)$

$y = \tanh^{-1} x$	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right);\ \lvert x\rvert < 1$

Proof: ($\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$ **)**

$$Recall:\ y = \sinh^{-1} x \Leftrightarrow x = \sinh y$$

To find its derivative we proceed implicitly:

- Given $x = \sinh y$. Differentiating w.r.t. x:

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sinh y)$$

$$1 = \cosh y \frac{dy}{dx}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cosh y}$$

- Since $-\infty < y < \infty$, $\cosh y \geq 0$, so using the identity

  $\cosh^2 y - \sinh^2 y = 1$:

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

$$\therefore \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$

➢ Other ways to obtain the derivatives are:

(a) $y = \sinh^{-1} x \Leftrightarrow x = \sinh y$ then

$x = \dfrac{e^y - e^{-y}}{2}$. Hence, find $\dfrac{dy}{dx}$.

(b) $y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$.

Hence, find $\dfrac{dy}{dx}$.

Standard Derivatives

Function, y	**Derivatives, $\frac{dy}{dx}$**
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}};\ \lvert x\rvert > 1$
$\tanh^{-1} x$	$\frac{1}{1-x^2};\ \lvert x\rvert < 1$
$\coth^{-1} x$	$\frac{1}{1-x^2};\ \lvert x\rvert > 1$
$\text{sech}^{-1} x$	$-\frac{1}{x\sqrt{1-x^2}};\ 0 < x < 1$
$\text{cosech}^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{1+x^2}};\ x \neq 0$

Generalised Form

$y = f(u);$ $u = g(x)$	$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$\sinh^{-1} u$	$\frac{1}{\sqrt{u^2+1}} \frac{du}{dx}$
$\cosh^{-1} u$	$\frac{1}{\sqrt{u^2-1}} \frac{du}{dx}; \ \lvert u \rvert > 1$
$\tanh^{-1} u$	$\frac{1}{1-u^2} \frac{du}{dx}; \ \lvert u \rvert < 1$
$\coth^{-1} u$	$\frac{1}{1-u^2} \frac{du}{dx}; \ \lvert u \rvert > 1$
$\text{sech}^{-1} u$	$-\frac{1}{u\sqrt{1-u^2}} \frac{du}{dx}; \ 0 < u < 1$
$\text{cosech}^{-1} u$	$\frac{1}{\lvert u \rvert \sqrt{1+u^2}} \frac{du}{dx}; \ u \neq 0$

Example 2.4:

Find the derivatives of

(a) $y = \sinh^{-1}(1-3x)$

(b) $y = \cosh^{-1}\left(\frac{1}{x}\right)$

(c) $y = e^x \operatorname{sech}^{-1} x$

(d) $y = \sinh^{-1}(\tan 3x)$

(e) $f(t) = \frac{\tanh^{-1} t^2}{1-\sec t}$

(f) $y = \sqrt{\coth^{-1} u}$

(g) $y = \cos\ 4x\ \cosh^{-1} 4x$

(h) $y^3 - \sinh^{-1} xy = 0$