Numerical Methods II

SSCM 3423

Chapter 3

This chapter solves boundary value problems (BVP) using finite difference methods (FDM).

Dr. Yeak Su Hoe
Department of Mathematical Sciences
Faculty of Science, Universiti Teknologi Malaysia
81300 UTM Johor Bahru, Malaysia
s.h.yeak@utm.my

June 2016

Ordinary differential equations (ODEs)

Finite difference method for linear second-order boundary value problem

General form: $y''+p(x)y'+q(x)y=r(x),\ a\le x\le b,\ y(a)=\alpha,\ y(b)=\beta.$ (a)

Centered-difference formula for second derivative:

$$y''(x_i)=\frac{1}{h^2}\left[y(x_{i+1})-2y(x_i)+y(x_{i-1})\right]-\frac{h^2}{12}y^{(4)}(\xi_i),\quad x_{i-1}<\xi_i<x_{i+1}.$$

Centered-difference formula for first derivative:

$$y'(x_i)=\frac{1}{2h}\left[y(x_{i+1})-y(x_{i-1})\right]-\frac{h^2}{6}y^{(3)}(\xi_i),\quad x_{i-1}<\xi_i<x_{i+1}.$$

Let divide the interval $[a,b]$ into N equal subintervals where $x_0=a$, $x_i=x_0+ih$, $\{i=1,2,\ldots,N\}$, $x_N=b$ and $h=(b-a)/N$.

At point $x=x_i$, equation (a) becomes

$$y_i''+p_iy_i'+q_iy_i=r_i,$$

Using centered-difference formula, we get

$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+p_i\frac{y_{i+1}-y_{i-1}}{2h}+q_iy_i=r_i,\ (\times h^2)\rightarrow(y_{i+1}-2y_i+y_{i-1})+p_i\frac{h}{2}(y_{i+1}-y_{i-1})+h^2q_iy_i=h^2r_i$$

$$\left(1-\frac{h}{2}p_i\right)y_{i-1}-\left(2-h^2q_i\right)y_i+\left(1+\frac{h}{2}p_i\right)y_{i+1}=h^2r_i.$$

For $i=1,2,\ldots,N-1$, the above equation will produce system $(N-1)$ equations with unknowns $y_0,y_1,\ldots,y_N$. With the given boundary condition, $y_0=\alpha$ and $y_N=\beta$, the system can be solved for $y_1,y_2,\ldots,y_{N-1}$ in matrix form, **Ay**=**b** (where matrix **A** is tridiagonal matrix with diagonally dominant, $|a_{ii}|>\sum|a_{ij}|,\ j=1\ldots n,\ j\neq n,$ row direction).

Tridiagonal system, **Ay**=**b**, can be solved using Thomas algorithm.

Ordinary differential equations (ODEs)

Finite difference method for linear second-order boundary value problem

Thomas algorithm

For tridiagonal system with size $n \times n$, **Ax**=**b**, matrix **A** can be factored into **A**=**LU**, where **L** (lower triangular Matrix) and **U**(upper triangular matrix) as below:

$$A = LU \rightarrow \begin{pmatrix} d_1 & e_1 & 0 & \cdots & 0 \\ c_2 & d_2 & e_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-1} & d_{n-1} & e_{n-1} \\ 0 & \cdots & 0 & c_n & d_n \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ c_2 & \alpha_2 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c_{n-1} & \alpha_{n-1} & 0 \\ 0 & \cdots & 0 & c_n & \alpha_n \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta_1 & 0 & \cdots & 0 \\ 0 & 1 & \beta_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & 1 & \beta_{n-1} \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

According to the above system, α_i and β_i can be calculated as

$\alpha_1=d_1$; $\alpha_i=d_i-c_i\beta_{i-1}$, $i=2,3,\ldots,n$; $\beta_i=e_i/\alpha_i$, $i=1,2,\ldots,n-1$.

The system **Ax**=**b** can be factorized as **LUx**=**b**, by letting **w**=**Ux**, we get **Lw**=**b**, then

(1) Solve **Lw**=**b** by forward substitution, we get

$$w_1=b_1/\alpha_1, \qquad w_i=(b_i-c_iw_{i-1})/\alpha_i,\ i=2,3,\ldots,n.$$

(2)Solve **Ux**=**w** by backward substitution, we get

$$x_n=w_n, \qquad x_i=w_i-\beta_ix_{i+1},\ i=n-1, n-2,\ldots,1.$$

The whole Thomas algorithm can be summarized as:

1. $\alpha_1=d_1$
2. $\alpha_i=d_i-c_i\beta_{i-1}$, $i=2,3,\ldots,n$
3. $\beta_i=e_i/\alpha_i$, $i=1,2,\ldots,n-1$.
4. $w_1=b_1/\alpha_1$
5. $w_i=(b_i-c_iw_{i-1})/\alpha_i$, $i=2,3,\ldots,n$.
6. $x_n=w_n$
7. $x_i=w_i-\beta_ix_{i+1}$, $i=n-1, n-2,\ldots,1$.

Ordinary differential equations (ODEs)

Finite difference method for linear second-order boundary value problem

Solve the linear boundary value problem

$$y''+(1/x)y'-(1/x^2)y=3, \quad y(1)=2, \ y(2)=3$$

for x=1(0.2)2 using finite difference method. Analytical solution: $y(x)=x(x-1)+2/x$.

Let h=0.2, $x_0=a=1$, $x_1=1.2$, $x_2=1.4$, $x_3=1.6$, $x_4=1.8$ and $x_5=b=2$. Find $y_i \approx y(x_i)$, i=1,2,3,4.

At x_i, we get

$$y_i''+\left(\frac{1}{x_i}\right)y_i'-\left(\frac{1}{x_i^2}\right)y_i=3 \quad\rightarrow\quad \left(\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right)+\left(\frac{1}{x_i}\right)\left(\frac{y_{i+1}-y_{i-1}}{2h}\right)-\left(\frac{1}{x_i^2}\right)y_i=3$$

Multiply with h^2, here we use 4 decimal place.

$$(y_{i+1}-2y_i+y_{i-1})+\frac{h}{2}\left(\frac{1}{x_i}\right)(y_{i+1}-y_{i-1})-h^2\left(\frac{1}{x_i^2}\right)y_i=3h^2 \quad\rightarrow\quad \left(1-\frac{0.1}{x_i}\right)y_{i-1}-\left[2+\left(\frac{0.2}{x_i}\right)^2\right]y_i+\left(1+\frac{0.1}{x_i}\right)y_{i+1}=0.12$$

For i=1, $$\left(1-\frac{0.1}{x_1}\right)y_0-\left[2+\left(\frac{0.2}{x_1}\right)^2\right]y_1+\left(1+\frac{0.1}{x_1}\right)y_2=0.12 \quad\rightarrow\quad \left(1-\frac{0.1}{1.2}\right)2-\left[2+\left(\frac{0.2}{1.2}\right)^2\right]y_1+\left(1+\frac{0.1}{1.2}\right)y_2=0.12$$

$$\rightarrow\quad -2.0278y_1+1.0833y_2=-1.7133$$

For i=2, $$\left(1-\frac{0.1}{x_2}\right)y_1-\left[2+\left(\frac{0.2}{x_2}\right)^2\right]y_2+\left(1+\frac{0.1}{x_2}\right)y_3=0.12 \quad\rightarrow\quad \left(1-\frac{0.1}{1.4}\right)y_1-\left[2+\left(\frac{0.2}{1.4}\right)^2\right]y_2+\left(1+\frac{0.1}{1.4}\right)y_3=0.12$$

$$\rightarrow\quad 0.9286y_1-2.0204y_2+1.0714y_3=0.12$$

For i=3, $$\left(1-\frac{0.1}{x_3}\right)y_2-\left[2+\left(\frac{0.2}{x_3}\right)^2\right]y_3+\left(1+\frac{0.1}{x_3}\right)y_4=0.12 \quad\rightarrow\quad \left(1-\frac{0.1}{1.6}\right)y_2-\left[2+\left(\frac{0.2}{1.6}\right)^2\right]y_3+\left(1+\frac{0.1}{1.6}\right)y_4=0.12$$

$$\rightarrow\quad 0.9375y_2-2.0156y_3+1.0625y_4=0.12$$

For i=4, $$\left(1-\frac{0.1}{x_4}\right)y_3-\left[2+\left(\frac{0.2}{x_4}\right)^2\right]y_4+\left(1+\frac{0.1}{x_4}\right)y_5=0.12 \quad\rightarrow\quad \left(1-\frac{0.1}{1.8}\right)y_3-\left[2+\left(\frac{0.2}{1.8}\right)^2\right]y_4+\left(1+\frac{0.1}{1.8}\right)(3)=0.12$$

$$\rightarrow\quad 0.9444y_3-2.0123y_4=-3.0468$$

Ordinary differential equations (ODEs)

Finite difference method for linear second-order boundary value problem

Finally, we get the tridiagonal system as below:

$$\mathbf{Ay}=\mathbf{b} \quad \rightarrow \begin{pmatrix} -2.0278 & 1.0833 & 0 & 0 \\ 0.9286 & -2.0204 & 1.0714 & 0 \\ 0 & 0.9375 & -2.0156 & 1.0625 \\ 0 & 0 & 0.9444 & -2.0123 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}=\begin{pmatrix} -1.7133 \\ 0.1200 \\ 0.1200 \\ -3.0468 \end{pmatrix}$$

Using Thomas algorithm, we get

i	**1**	**2**	**3**	**4**
d_i e_i c_i b_i	-2.0278 - 1.0833 -1.7133	-2.0204 0.9286 1.0714 0.1200	-2.0156 0.9375 1.0625 0.1200	-2.0123 0.9444 - -3.0468
$(\alpha_1=d_1)$ $\alpha_i=d_i-c_i\beta_{i-1}$,	-2.0278	-1.5243	-1.3566	-1.2726
$\beta_i=e_i/\alpha_i$,	-0.5342	-0.7029	-0.7832	-
$(w_1=b_1/\alpha_1)$ $w_i=(b_i-c_iw_{i-1})/\alpha_i$,	0.8449	0.4360	0.2128	2.5521
$(y_n=w_n)$ $y_i=w_i-\beta_i y_{i+1}$,	1.9082	1.9905	2.2116	2.5521

Finally, we get $y(1.2)\approx y_1=1.9082$, $y_2=1.9905$, $y_3=2.2116$ and $y(1.8)\approx y_4=2.5521$.
The exact solution is given as $y(1.2)=1.9067$, $y(1.4)=1.9886$, $y(1.6)=2.2100$, $y(1.8)=2.5511$.
So, finite difference method produce results accurate up to 2 decimal places.

Ordinary differential equations (ODEs)

Finite difference method for nonlinear second-order boundary value problem

For the general nonlinear boundary value problem

$$y''=f(x,y,y'), \qquad a\le x\le b,\ y(a)=\alpha,\ y(b)=\beta, \qquad (a)$$

Let divide the interval $[a,b]$ into N equal subintervals where $x_0=a$, $x_i=x_0+ih$, $\{i=1,2,\ldots,N\}$, $x_N=b$ and $h=(b-a)/N$.

At point $x=x_i$, equation (a) becomes

$$-\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+f\left(x_i,y_i,\frac{y_{i+1}-y_{i-1}}{2h}\right)=0 \;\rightarrow\; \boxed{-y_{i-1}+2y_i-y_{i+1}+h^2 f\left(x_i,y_i,\frac{y_{i+1}-y_{i-1}}{2h}\right)=0=f_i\left(y_1,\cdots,y_{N-1}\right)}$$

For $i=1,2,\ldots,N-1$, the above equation will produce nonlinear system $(N-1)$ equations with unknowns $y_0,y_1,\ldots,y_N$. The above nonlinear system has a unique solution if $h<2/L$, $L=\max|f_{y'}(x,y,y')|$. With the given boundary condition, $y_0=\alpha$ and $y_N=\beta$, the system can be solved by Newton's method for nonlinear systems. A sequence of iteration will converge to solution if the guess initial approximation is sufficiently close to solution.

The Jacobian matrix, $J(y_1,\ldots,y_{N-1})$ is tridiagonal with ij-th entry:

$$J\left(y_1,\cdots,y_{N-1}\right)_{ij}=\begin{cases}-1+\dfrac{h}{2}f_{y'}\left(x_i,y_i,\dfrac{y_{i+1}-y_{i-1}}{2h}\right), & \text{for } i=j-1 \text{ and } j=2,\cdots,N-1,\\[2ex] 2+h^2 f_y\left(x_i,y_i,\dfrac{y_{i+1}-y_{i-1}}{2h}\right), & \text{for } i=j \text{ and } j=1,\cdots,N-1,\\[2ex] -1-\dfrac{h}{2}f_{y'}\left(x_i,y_i,\dfrac{y_{i+1}-y_{i-1}}{2h}\right), & \text{for } i=j+1 \text{ and } j=1,\cdots,N-2.\end{cases}$$

Correction vector can be calculated using Thomas algorithm:

$$\boxed{J\cdot\begin{bmatrix}h_1^{(k)}\\ \vdots\\ h_{N-1}^{(k)}\end{bmatrix}=-\begin{bmatrix}f_1\left(y_1,\cdots,y_{N-1}\right)\\ \vdots\\ f_{N-1}\left(y_1,\cdots,y_{N-1}\right)\end{bmatrix} \;\rightarrow\; \begin{bmatrix}y_1^{(k+1)}\\ \vdots\\ y_{N-1}^{(k+1)}\end{bmatrix}=\begin{bmatrix}y_1^{(k)}\\ \vdots\\ y_{N-1}^{(k)}\end{bmatrix}+\begin{bmatrix}h_1^{(k)}\\ \vdots\\ h_{N-1}^{(k)}\end{bmatrix}.}$$

The Newton iteration will stop when the solutions converge to certain decimal places or some norm stopping criteria.

Ordinary differential equations (ODEs)

Finite difference method for nonlinear second-order boundary value problem

Newton's method for nonlinear systems

$$\begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

The system of equations $g_i(y_1,y_2,\ldots,y_n)=0 \quad (1\le i\le n)$

can be expressed simply as $\mathbf{G}(\mathbf{Y})=\mathbf{0}$

by letting $\mathbf{Y}=(y_1,y_2,\ldots,y_n)^T$ and $\mathbf{G}=(g_1,g_2,\ldots,g_n)^T$. Using the Taylor's series expansion, we get

$$\mathbf{0}=\mathbf{G}(\mathbf{Y}+\mathbf{H}) \approx \mathbf{G}(\mathbf{Y})+\mathbf{G}'(\mathbf{Y})\mathbf{H}\ ,$$ *(where* $\mathbf{Y}+\mathbf{H}$ *is more accurate solution)*

where $\mathbf{H}=(h_1,h_2,\ldots,h_n)^T$ and $\mathbf{G}'(\mathbf{Y})$ is the $n\times n$ Jacobian matrix $\mathbf{J}(\mathbf{Y})$:

$$\mathbf{J}(\mathbf{Y})=\mathbf{G}'(\mathbf{Y})=\begin{bmatrix} \partial g_1/\partial y_1 & \partial g_1/\partial y_2 & \cdots & \partial g_1/\partial y_n \\ \partial g_2/\partial y_1 & \partial g_2/\partial y_2 & \cdots & \partial g_2/\partial y_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial g_n/\partial y_1 & \partial g_n/\partial y_2 & \cdots & \partial g_n/\partial y_n \end{bmatrix}$$

$y'' + (y')^3 y = 0$

$ans: y^3/3 - 2c_1 y = 2x + c2$

$let\ c_1 = c_2 = 0$

$y^3 = 6x$

$x = 1(0.25)2$

The correction vector $\mathbf{H}$ is obtained by solving linear system

$$\mathbf{J}(\mathbf{Y})\mathbf{H}= -\mathbf{G}(\mathbf{Y})$$

If Jacobian matrix is tridiagonal matrix, then $\mathbf{H}$ can be solved using Thomas algorithm. If the matrix size is 2×2, then just use the inverse of matrix $\mathbf{J}$, $\mathbf{H}=\mathbf{J}^{-1}(-\mathbf{G})$.

Finally, Newton's iteration for n nonlinear equations in n variables is given by

$$\mathbf{Y}^{(k+1)}=\mathbf{Y}^{(k)}+\mathbf{H}^{(k)} \quad \rightarrow \mathbf{Y}^{(k+1)}=\mathbf{Y}^{(k)}-\mathbf{J}^{-1}\mathbf{G}$$

where the Jacobian system is

$$\mathbf{J}(\mathbf{Y}^{(k)})\mathbf{H}^{(k)}= -\mathbf{G}(\mathbf{Y}^{(k)}).$$

Ordinary differential equations (ODEs)

Finite difference method for nonlinear second-order boundary value problem

E.g. use nonlinear finite difference method to solve boundary value problem

$$y'' = -(y')^2 - y + \ln x,\quad 1 \le x \le 2,\ y(1)=0,\ y(2)=\ln 2=0.6931.$$

for x=1(0.2)2. Analytical solution: $y=\ln x$. (use 4 decimal places). Stopping criterion: Tolerance, ε=0.02 using *maximum-magnitude* norm.

Let h=0.2, $x_0=a=1$, $x_1=1.2$, $x_2=1.4$, $x_3=1.6$, $x_4=1.8$ and $x_5=b=2$. Find $y_i \approx y(x_i)$, i=1,2,3,4. N=5.

At x_i, we get

$$-\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+f\left(x_i, y_i, \frac{y_{i+1}-y_{i-1}}{2h}\right)=0 \quad\rightarrow\quad \boxed{-y_{i-1}+2y_i-y_{i+1}+h^2\left(-\left(\frac{y_{i+1}-y_{i-1}}{2h}\right)^2-y_i+\ln x_i\right)=0=g_i}$$

For i=1, $-y_0+2y_1-y_2+0.2^2\left(-\left(\frac{y_2-y_0}{2(0.2)}\right)^2-y_1+\ln x_1\right)=0 \rightarrow -0+2y_1-y_2+\left(-\frac{1}{4}(y_2-0)^2-0.2^2y_1+0.2^2\cdot 0.1823\right)=0=g_1$

For i=2, $-y_1+2y_2-y_3+0.2^2\left(-\left(\frac{y_3-y_1}{2(0.2)}\right)^2-y_2+\ln x_2\right)=0 \rightarrow -y_1+2y_2-y_3+\left(-\frac{1}{4}(y_3-y_1)^2-0.2^2y_2+0.2^2\cdot 0.3365\right)=0=g_2$

For i=3, $-y_2+2y_3-y_4+0.2^2\left(-\left(\frac{y_4-y_2}{2(0.2)}\right)^2-y_3+\ln x_3\right)=0 \rightarrow -y_2+2y_3-y_4+\left(-\frac{1}{4}(y_4-y_2)^2-0.2^2y_3+0.2^2\cdot 0.4700\right)=0=g_3$

For i=4, $-y_3+2y_4-y_5+0.2^2\left(-\left(\frac{y_5-y_3}{2(0.2)}\right)^2-y_4+\ln x_4\right)=0 \rightarrow -y_3+2y_4-0.6931+\left(-\frac{1}{4}(0.6931-y_3)^2-0.2^2y_4+0.2^2\cdot 0.5878\right)=0=g_4$

$$\mathbf{J}(\mathbf{Y})=\begin{bmatrix}\partial g_1/\partial y_1 & \partial g_1/\partial y_2 & \cdots & \partial g_1/\partial y_4\\ \partial g_2/\partial y_1 & \partial g_2/\partial y_2 & \cdots & \partial g_2/\partial y_4\\ \partial g_3/\partial y_1 & \partial g_3/\partial y_2 & \ddots & \vdots\\ \partial g_4/\partial y_1 & \partial g_4/\partial y_2 & \cdots & \partial g_4/\partial y_4\end{bmatrix}=\begin{bmatrix}2-0.2^2 & -1-\tfrac{1}{2}(y_2-\mathbf{0}) & 0 & 0\\ -1-\tfrac{1}{2}(y_3-y_1)(-1) & 2-0.2^2 & -1-\tfrac{1}{2}(y_3-y_1) & 0\\ 0 & -1-\tfrac{1}{2}(y_4-y_2)(-1) & 2-0.2^2 & -1-\tfrac{1}{2}(y_4-y_2)\\ 0 & 0 & -1-\tfrac{1}{2}(\mathbf{0.6931}-y_3)(-1) & 2-0.2^2\end{bmatrix}$$

UTM

Ordinary differential equations (ODEs)

Finite difference method for nonlinear second-order boundary value problem

$$\mathbf{J}(\mathbf{Y})=\begin{bmatrix}1.96 & -1-\tfrac{1}{2}y_2 & 0 & 0\\ -1+\tfrac{1}{2}(y_3-y_1) & 1.96 & -1-\tfrac{1}{2}(y_3-y_1) & 0\\ 0 & -1+\tfrac{1}{2}(y_4-y_2) & 1.96 & -1-\tfrac{1}{2}(y_4-y_2)\\ 0 & 0 & -0.6534-\tfrac{1}{2}y_3 & 1.96\end{bmatrix}$$

To guess the initial values, we use linear interpolation, h=(ln2-0)/5≈0.14; where y_0=0, y_5=0.7. So, we get

$$\mathbf{Y}^{(0)}=\begin{bmatrix}y_1^{(0)}\\ y_2^{(0)}\\ y_3^{(0)}\\ y_4^{(0)}\end{bmatrix}=\begin{bmatrix}0.14\\0.28\\0.42\\0.56\end{bmatrix},\ \mathbf{J}\left(\mathbf{Y}^{(0)}\right)=\begin{bmatrix}1.96 & -1-\tfrac{1}{2}0.28 & 0 & 0\\ -1+\tfrac{1}{2}(0.42-0.14) & 1.96 & -1-\tfrac{1}{2}(0.42-0.14) & 0\\ 0 & -1+\tfrac{1}{2}(0.56-0.28) & 1.96 & -1-\tfrac{1}{2}(0.56-0.28)\\ 0 & 0 & -0.6534-\tfrac{1}{2}0.42 & 1.96\end{bmatrix}$$

$$\rightarrow \mathbf{J}\left(\mathbf{Y}^{(0)}\right)=\begin{bmatrix}1.96 & -1.14 & 0 & 0\\ -0.86 & 1.96 & -1.14 & 0\\ 0 & -0.86 & 1.96 & -1.14\\ 0 & 0 & -0.8634 & 1.96\end{bmatrix},\quad -\mathbf{G}\left(\mathbf{Y}^{(0)}\right)=-\begin{bmatrix}2(0.14)-0.28-\tfrac{1}{4}0.28^2-0.2^2 0.14+0.0073\\ -0.14+2\cdot 0.28-0.42-\tfrac{1}{4}(0.42-0.14)^2-0.2^2 0.28+0.0135\\ -0.28+2\cdot 0.42-0.56-\tfrac{1}{4}(0.56-0.28)^2-0.2^2 0.42+0.0188\\ -0.42+2\cdot 0.56-0.6931-\tfrac{1}{4}(0.6931-0.42)^2-0.2^2 0.56+0.0235\end{bmatrix}=-\begin{bmatrix}-0.0179\\-0.0173\\-0.0176\\-0.0106\end{bmatrix}$$

$$\begin{bmatrix}1.96 & -1.14 & 0 & 0\\ -0.86 & 1.96 & -1.14 & 0\\ 0 & -0.86 & 1.96 & -1.14\\ 0 & 0 & -0.8634 & 1.96\end{bmatrix}\begin{bmatrix}h_1^{(0)}\\h_2^{(0)}\\h_3^{(0)}\\h_4^{(0)}\end{bmatrix}=-\begin{bmatrix}-0.0179\\-0.0173\\-0.0176\\-0.0106\end{bmatrix}\rightarrow \text{Thomas algorithm}\rightarrow \begin{bmatrix}h_1^{(0)}\\h_2^{(0)}\\h_3^{(0)}\\h_4^{(0)}\end{bmatrix}=\begin{bmatrix}0.0414\\0.0556\\0.0491\\0.0270\end{bmatrix}.$$

$$\begin{bmatrix}y_1^{(1)}\\y_2^{(1)}\\y_3^{(1)}\\y_4^{(1)}\end{bmatrix}=\begin{bmatrix}y_1^{(0)}\\y_2^{(0)}\\y_3^{(0)}\\y_4^{(0)}\end{bmatrix}+\begin{bmatrix}h_1^{(0)}\\h_2^{(0)}\\h_3^{(0)}\\h_4^{(0)}\end{bmatrix}=\begin{bmatrix}0.14\\0.28\\0.42\\0.56\end{bmatrix}+\begin{bmatrix}0.0414\\0.0556\\0.0491\\0.0270\end{bmatrix}=\begin{bmatrix}0.1814\\0.3356\\0.4691\\0.587\end{bmatrix}.$$

Ordinary differential equations (ODEs)

Finite difference method for nonlinear second-order boundary value problem

$$\mathbf{Y}^{(1)}=\begin{bmatrix} y_1^{(1)} \\ y_2^{(1)} \\ y_3^{(1)} \\ y_4^{(1)} \end{bmatrix}=\begin{bmatrix} 0.1814 \\ 0.3356 \\ 0.4691 \\ 0.587 \end{bmatrix}, \mathbf{J}\left(\mathbf{Y}^{(1)}\right)=\begin{bmatrix} 1.96 & -1-\tfrac{1}{2}0.3356 & 0 & 0 \\ -1+\tfrac{1}{2}(0.4691-0.1814) & 1.96 & -1-\tfrac{1}{2}(0.4691-0.1814) & 0 \\ 0 & -1+\tfrac{1}{2}(0.587-0.3356) & 1.96 & -1-\tfrac{1}{2}(0.587-0.3356) \\ 0 & 0 & -0.6534-\tfrac{1}{2}0.4691 & 1.96 \end{bmatrix}$$

$$\rightarrow \mathbf{J}\left(\mathbf{Y}^{(1)}\right)=\begin{bmatrix} 1.96 & -1.1678 & 0 & 0 \\ -0.8562 & 1.96 & -1.1439 & 0 \\ 0 & -0.8743 & 1.96 & -1.1257 \\ 0 & 0 & -0.8880 & 1.96 \end{bmatrix}, \quad -\mathbf{G}\left(\mathbf{Y}^{(1)}\right)=-\begin{bmatrix} -0.0009 \\ 0.00004 \\ -0.00016 \\ -0.0007 \end{bmatrix}$$

$$\begin{bmatrix} 1.96 & -1.1678 & 0 & 0 \\ -0.8562 & 1.96 & -1.1439 & 0 \\ 0 & -0.8743 & 1.96 & -1.1257 \\ 0 & 0 & -0.8880 & 1.96 \end{bmatrix}\begin{bmatrix} h_1^{(1)} \\ h_2^{(1)} \\ h_3^{(1)} \\ h_4^{(1)} \end{bmatrix}=-\begin{bmatrix} -0.0009 \\ 0.00004 \\ -0.00016 \\ -0.0007 \end{bmatrix} \rightarrow \text{Thomas algorithm} \rightarrow \begin{bmatrix} h_1^{(1)} \\ h_2^{(1)} \\ h_3^{(1)} \\ h_4^{(1)} \end{bmatrix}=\begin{bmatrix} 0.0011 \\ 0.0010 \\ 0.0010 \\ 0.0008 \end{bmatrix}.$$

$$\begin{bmatrix} y_1^{(2)} \\ y_2^{(2)} \\ y_3^{(2)} \\ y_4^{(2)} \end{bmatrix}=\begin{bmatrix} y_1^{(1)} \\ y_2^{(1)} \\ y_3^{(1)} \\ y_4^{(1)} \end{bmatrix}+\begin{bmatrix} h_1^{(1)} \\ h_2^{(1)} \\ h_3^{(1)} \\ h_4^{(1)} \end{bmatrix}=\begin{bmatrix} 0.1814 \\ 0.3356 \\ 0.4691 \\ 0.587 \end{bmatrix}+\begin{bmatrix} 0.0011 \\ 0.0010 \\ 0.0010 \\ 0.0008 \end{bmatrix}=\begin{bmatrix} 0.1825 \\ 0.3366 \\ 0.4701 \\ 0.5878 \end{bmatrix}.$$

Stopping criterion: *maximum-magnitude* norm of increment solution vector < ε=0.02. $\left\|\mathbf{Y}^{(2)}-\mathbf{Y}^{(1)}\right\|_\infty=\left\|\mathbf{H}^{(1)}\right\|_\infty=0.0011<\varepsilon$.

The final solution is

$$\begin{bmatrix} y_1^{(2)} \\ y_2^{(2)} \\ y_3^{(2)} \\ y_4^{(2)} \end{bmatrix}=\begin{bmatrix} 0.1825 \\ 0.3366 \\ 0.4701 \\ 0.5878 \end{bmatrix}. \quad \textit{exact solution}: \begin{bmatrix} \ln 1.2 \\ \ln 1.4 \\ \ln 1.6 \\ \ln 1.8 \end{bmatrix}=\begin{bmatrix} 0.1823 \\ 0.3365 \\ 0.4700 \\ 0.5878 \end{bmatrix}.$$

Note: If the problem is simplified by only finding 2 points (y_1 and y_2), Then Thomas algorithm is not required since the matrix is 2×2. Use the below simple formula.

$$\mathbf{J}^{-1}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \rightarrow \quad \mathbf{H}=\mathbf{J}^{-1}(-\mathbf{G}).$$

References

- Brian Bradie, A Friendly Introduction to Numerical Analysis, Prentice Hall, New Jersey, 2006.
- Fausett L.V, Numerical Methods; algorithm and applications, Prentice Hall, New Jersey,2003
- Rao S.S, Applied Numerical Methods for Engineers and Scientist, Prentice Hall, New Jersey,2002
- Faires J.D. Burden R, Numerical Methods, 2nd Edition, Thomson Brooks/Cole, Australia,1998
- Burden R.L, Faires J.D & Reynolds A.C, Numerical Analysis, 5th edition, PWS-KENT Pub, Boston, 1993