CHAPTER 5
Response and Stability Analysis in Frequency Domain

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Chapter Outline

5.1	• Introduction to the concept of frequency response
5.2	• Introduction to simple open loop Bode plot
5.3	• Bode Plot - overall plotting
5.4	• Analysis of Bode Plot

5.1 Introduction to the concept of frequency response

The Concept

- Sinusoidal inputs to a linear system generate sinusoidal responses of the same frequency.
- However, they differ in amplitude and phase angle from the input.
- Sinusoids can be represented as complex numbers called *phasor*.

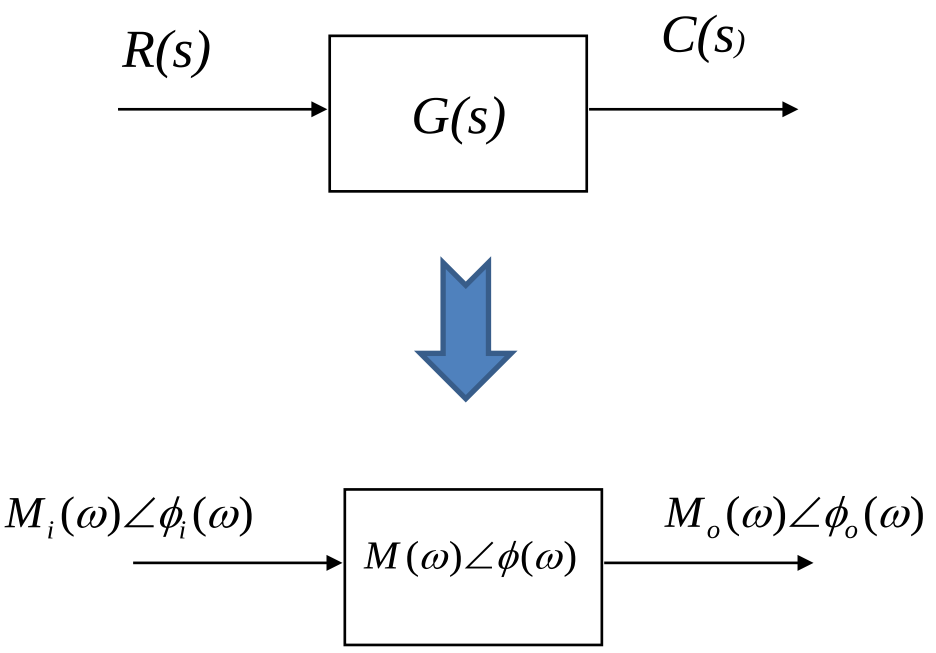
$$M_1 \cos(\omega t + \phi_1) = M_1 \angle \phi_1$$

where M_1 and ϕ_1 are the amplitude and phase angle.

- Thus, the system can also be represented by a complex number so that the product of the input phasor and the system yield the phasor of the output.

The Concept [2]

- The block diagram:



- The output sinusoid is found by multiplying the input and the system.

The Concept [3]

Thus $$M_o(\omega)\angle\phi_o(\omega) = M_i(\omega)M(\omega)\angle[\phi_i(\omega)+\phi(\omega)]$$

Magnitude frequency response $$M(w) = \frac{M_o(w)}{M_i(w)}$$

Phase frequency response $$\phi(\omega) = \phi_o(\omega) - \phi_i(\omega)$$

- The combination of the magnitude and phase frequency responses is called the frequency response,

$$M(\omega)\angle\phi(\omega)$$

- The frequency response of a system with transfer function $G(s)$ is $$G(j\omega) = G(s)\big|_{s\to j\omega}$$

Plotting Frequency Response

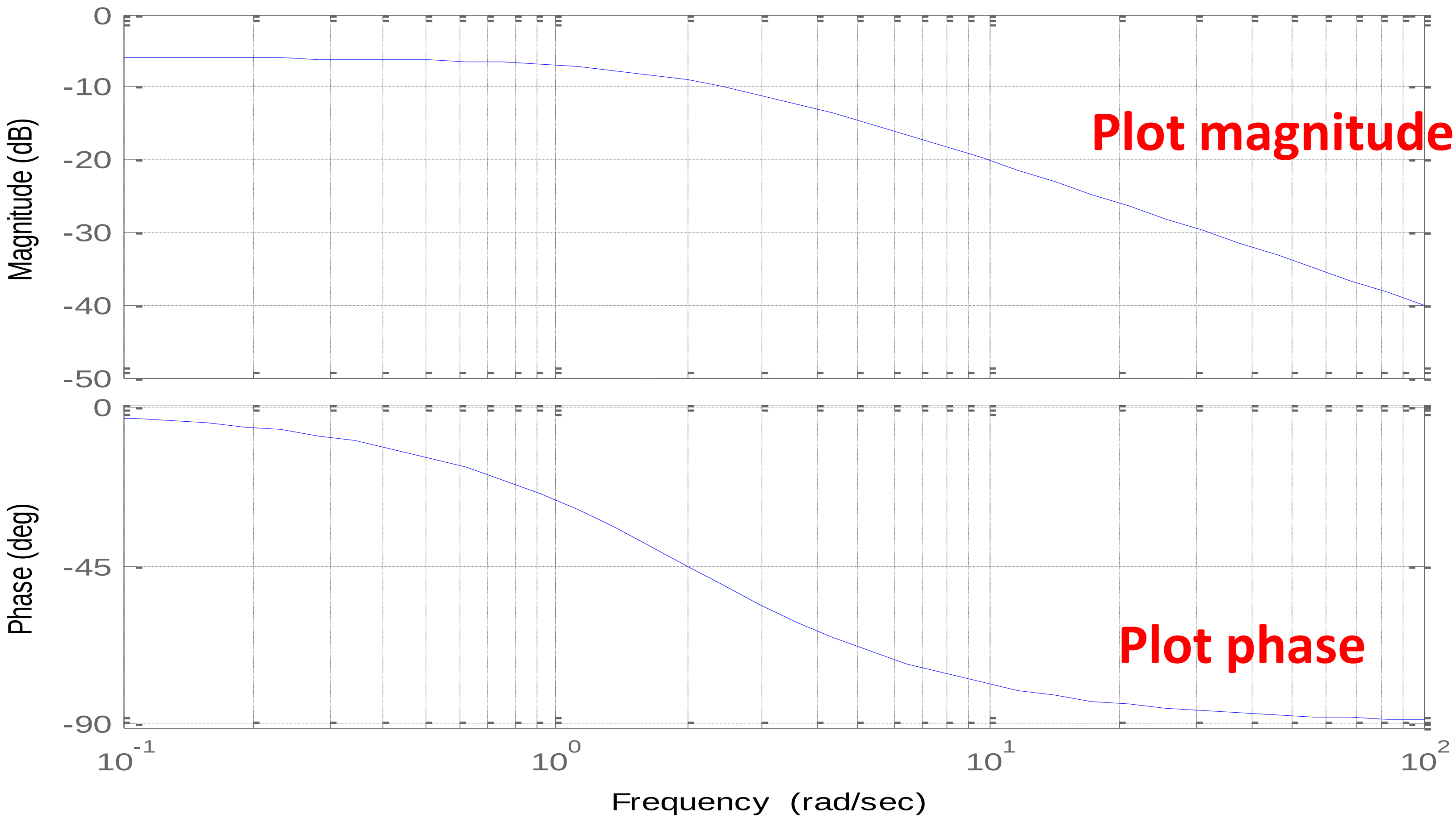
- For a given transfer function: G(s) = 1/(s+2)

$$G(j\omega) = \frac{1}{(j\omega + 2)}$$

$$|G(j\omega)| = \frac{1}{\sqrt{(\omega^2 + 4)}}; \quad \phi(\omega) = -\tan^{-1}(\omega/2)$$

- One way to plot the Frequency response is by using a separate magnitude and phase plots.
  a) Magnitude curve: decibel (dB) vs log ω [dB = 20 log M]
  b) Phase curve: *phase angle* vs *log* ω

- Separate plots (magnitude and phase vs log ω).



5.2 Introduction to simple open loop Bode plot

Bode Plots

- The log-magnitude and phase frequency response curves as a function of log ω is called Bode plots.
- Bode plot is a technique for analyses and design of control systems.
- Consider a transfer function $$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_k)}{s^m(s+p_1)(s+p_2)\cdots(s+p_n)}$$
- The magnitude frequency response

$$|G(j\omega)| = \left.\frac{K|(s+z_1)||(s+z_2)|\cdots|(s+z_k)|}{|s^m||(s+p_1)||(s+p_2)|\cdots|(s+p_n)|}\right|_{s\to j\omega}$$

- Converting into dB

$$20\log|G(j\omega)| = 20\log K + 20\log|(j\omega+z_1)| + 20\log|(j\omega+z_2)| + \cdots$$
$$-20\log|(j\omega)^m| - 20\log|(j\omega+p_1)| - 20\log|(j\omega+p_2)| - \cdots$$

Bode Plots

- The phase frequency response

$$\angle G(j\omega) = \angle K + \angle(j\omega + z_1) + \angle(j\omega + z_2) + \cdots - \angle(j\omega + p_1) - \angle(j\omega + p_2) - \cdots$$

- If we know the magnitude and phase responses of each term, total frequency response can be obtained by algebraic sum of each term.
- The frequency response can be simplified by utilizing straight-line approximations.
- Therefore, total frequency response can be obtained by graphic addition.

5.3
Bode Plot – overall plotting

Bode Plot

- Constant K : $G(s)=K$
- Zeros at origin: $G(s)=s$
- Poles at origin: $G(s)=1/s$
- Zeros at real-axis (s-plane): $G(s)=(s+a)$
- Poles at real axis (s-plane): $G(s)= 1/(s+b)$

Bode Plots - Constant *K*

- ***G(s) = K***

$$G(s) = K; G(j\omega) = K$$

$$\left| Gj\omega) \right|_{dB} = 20\log K$$

$$\angle G(j\omega) = 0^0$$

Bode Plots - Zeros at origin

- ***G(s) = s*** **(Zero at origin)**

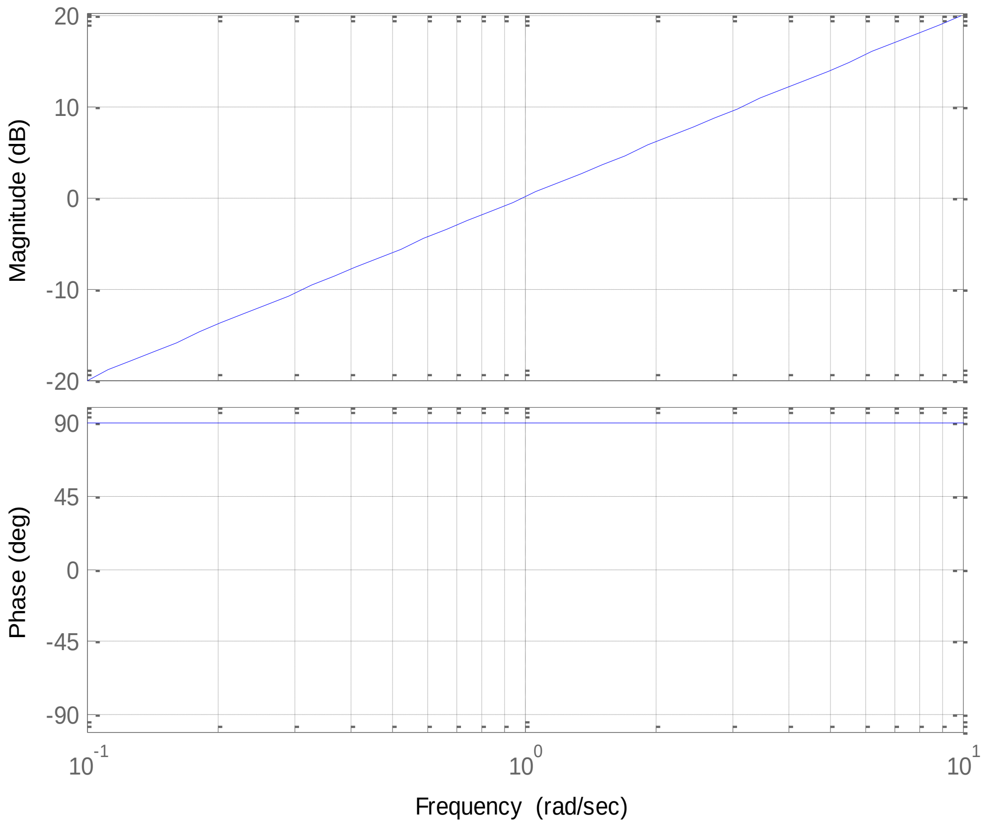
$$G(s) = s; G(j\omega) = j\omega$$

$$|Gj\omega)|_{dB} = 20\log\omega$$

$$\angle G(j\omega) = 90^0$$

- At ω = 1, gain = 0 dB.

20 dB/decade

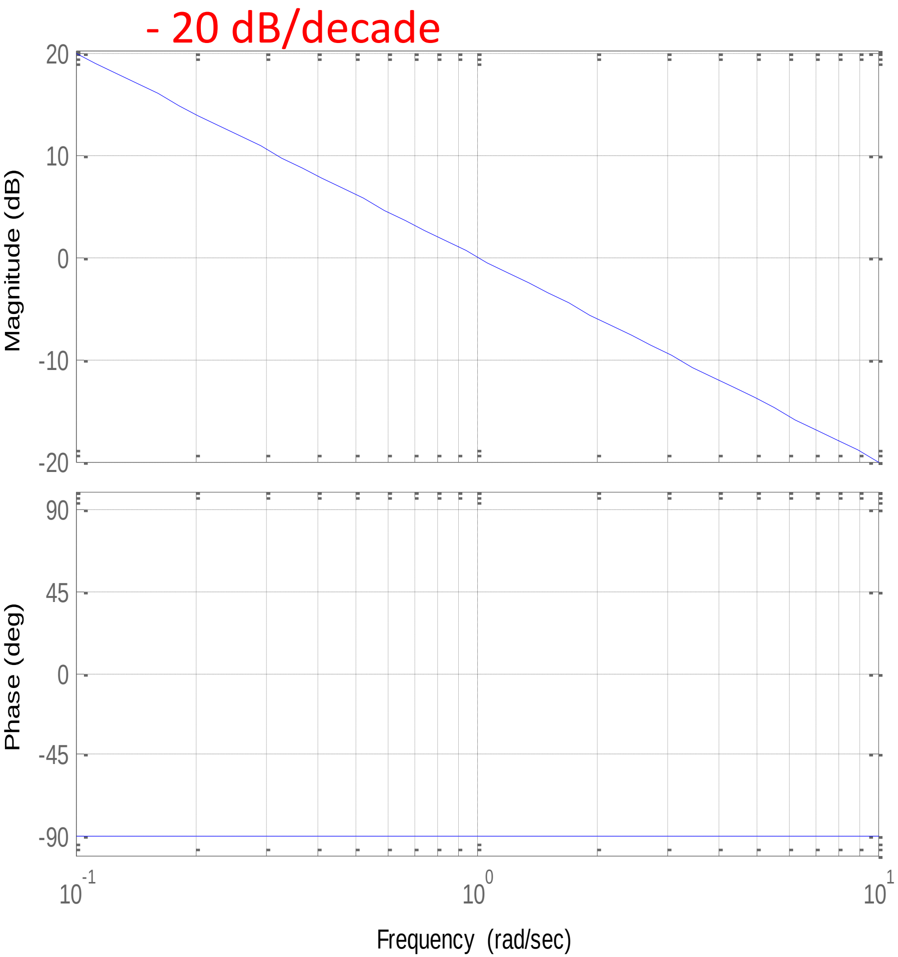


Bode Plots- Poles at origin

- ***G(s) = 1/s*** **(pole at origin)**

$$G(s) = 1/s ; G(j\omega) = 1/j\omega$$
$$|Gj\omega)| = \log(1/\omega)$$
$$|Gj\omega)|_{dB} = 20\log(1/\omega) = -20\log\omega$$
$$\angle G(j\omega) = -90^0$$

- At ω = 1, gain = 0 dB.



Bode Plots – Zeros at real axis

- For $G(s) = \frac{s}{a} + 1$ → $G(s) = (\frac{s}{a} + 1); G(j\omega) = \frac{j\omega}{a} + 1$

$$|Gj\omega)| = \sqrt{\left(\frac{\omega}{a}\right)^2 + 1}$$

$$\angle G(j\omega) = \tan^{-1}\left(\frac{\omega}{a}\right)$$

- At ω << a,

$$|Gj\omega)|_{dB} = 20\log 1 = 0dB$$

$$\angle G(j\omega) = \tan^{-1} 0 = 0^0$$

- At ω = a,

$$|Gj\omega)|_{dB} = 20\log\sqrt{2} = 3.01dB$$

$$\angle G(j\omega) = \tan^{-1} 1 = 45^0$$

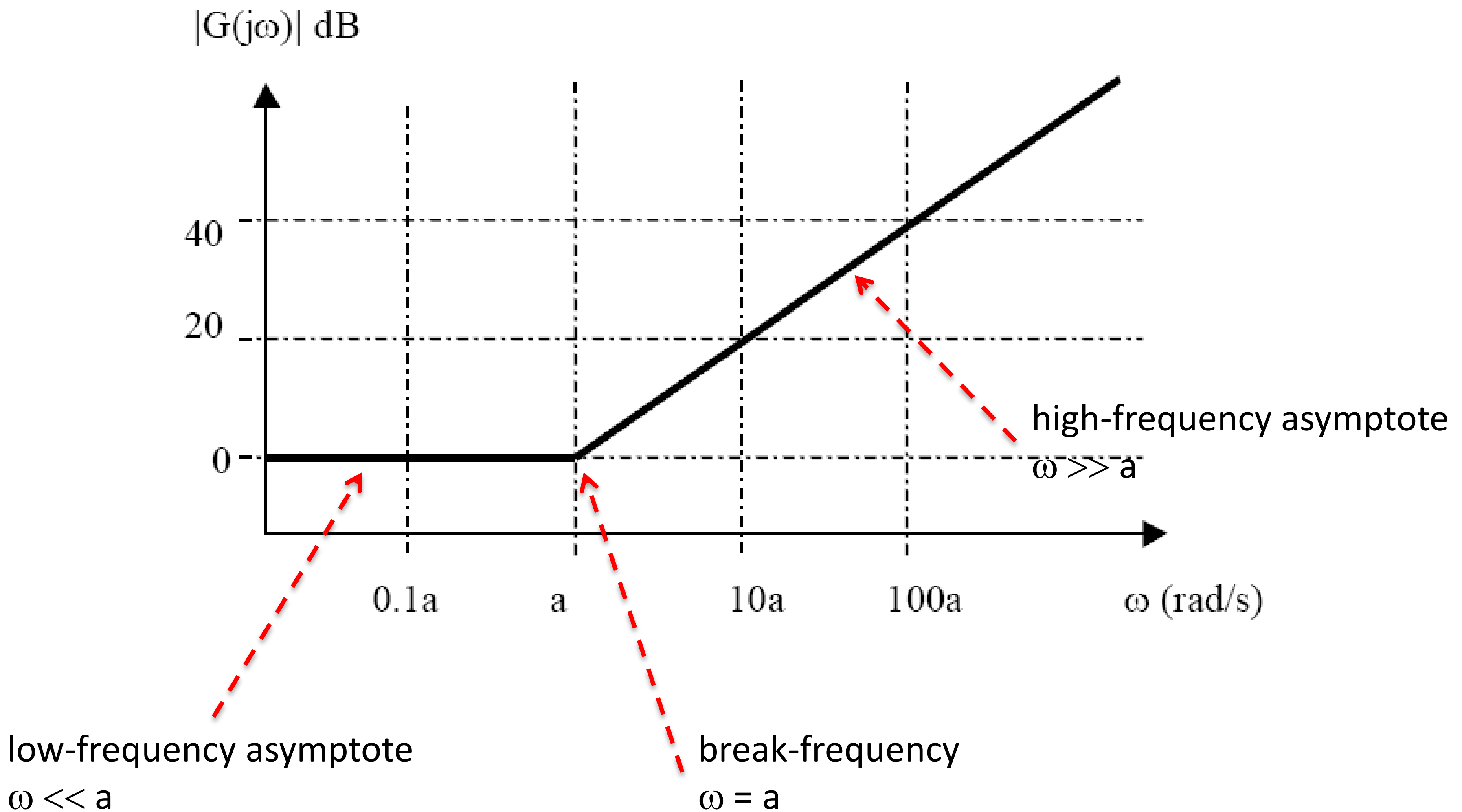
- At ω >> a,

$$|Gj\omega)|_{dB} = 20\log\omega$$ 20 dB/decade

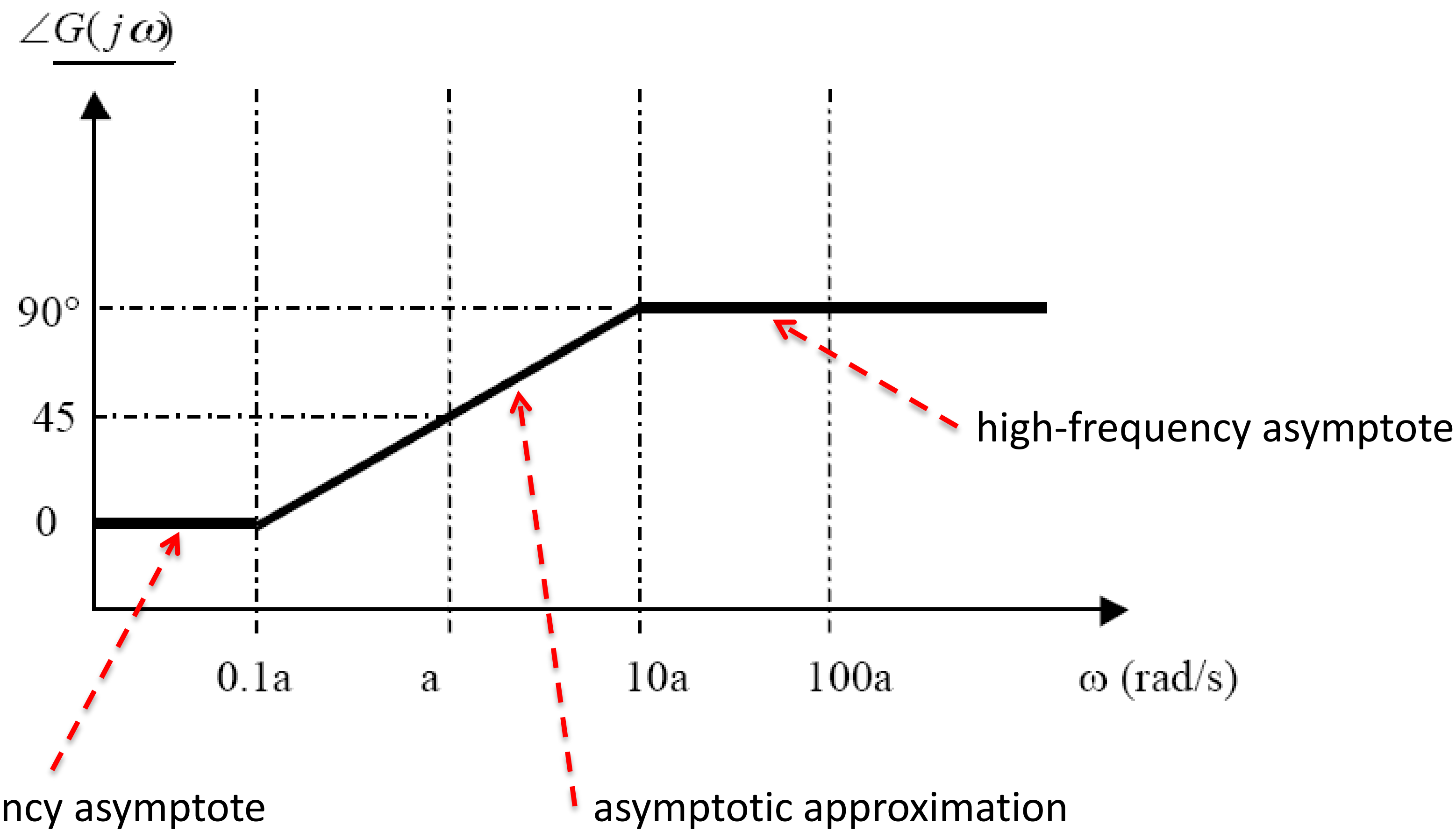
$$\angle G(j\omega) = \tan^{-1}\omega = 90^0$$

- The low-frequency approximation is called *the low-frequency asymptote*.
- The frequency, $\omega = a$ is known as *break frequency* because it is the break between the low- and high-frequency asymptotes.
- The high-frequency approximation is called the *high-frequency asymptote*.

Bode Plot- magnitude plot (Zeros)



Bode Plot- Phase plot (Zeros)



Bode Plots – Poles at real axis

- For $G(s)=\dfrac{1}{\frac{s}{a}+1}$ → $G(s)=\left(\dfrac{1}{s/a+1}\right); G(j\omega)=\left(\dfrac{1}{j\omega/a+1}\right)$

$$|Gj\omega)|=\frac{1}{\sqrt{\left(\frac{\omega}{a}\right)^2+1}};\angle G(j\omega)=-\tan^{-1}\left(\frac{\omega}{a}\right)$$

- At ω << a,

$$|Gj\omega)|_{dB}=20\log 1=0dB$$

$$\angle G(j\omega)=\tan^{-1}0=0^0$$

- At ω = a,

$$|Gj\omega)|_{dB}=20\log(1/\sqrt{2})=-3.01dB$$

$$\angle G(j\omega)=-\tan^{-1}1=-45^0$$

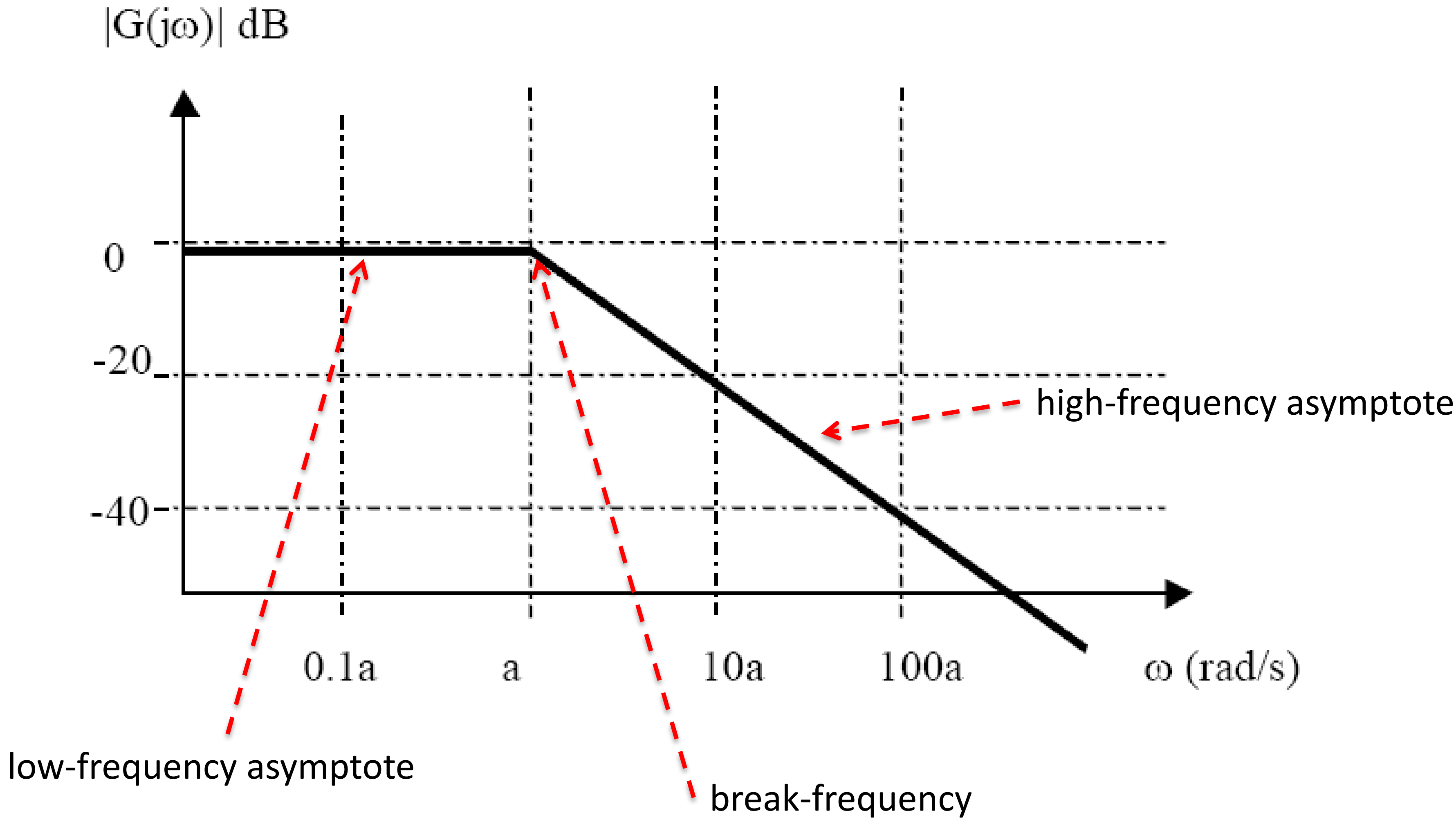
- At ω >> a,

$$|Gj\omega)|_{dB}=20\log\frac{1}{\omega}$$

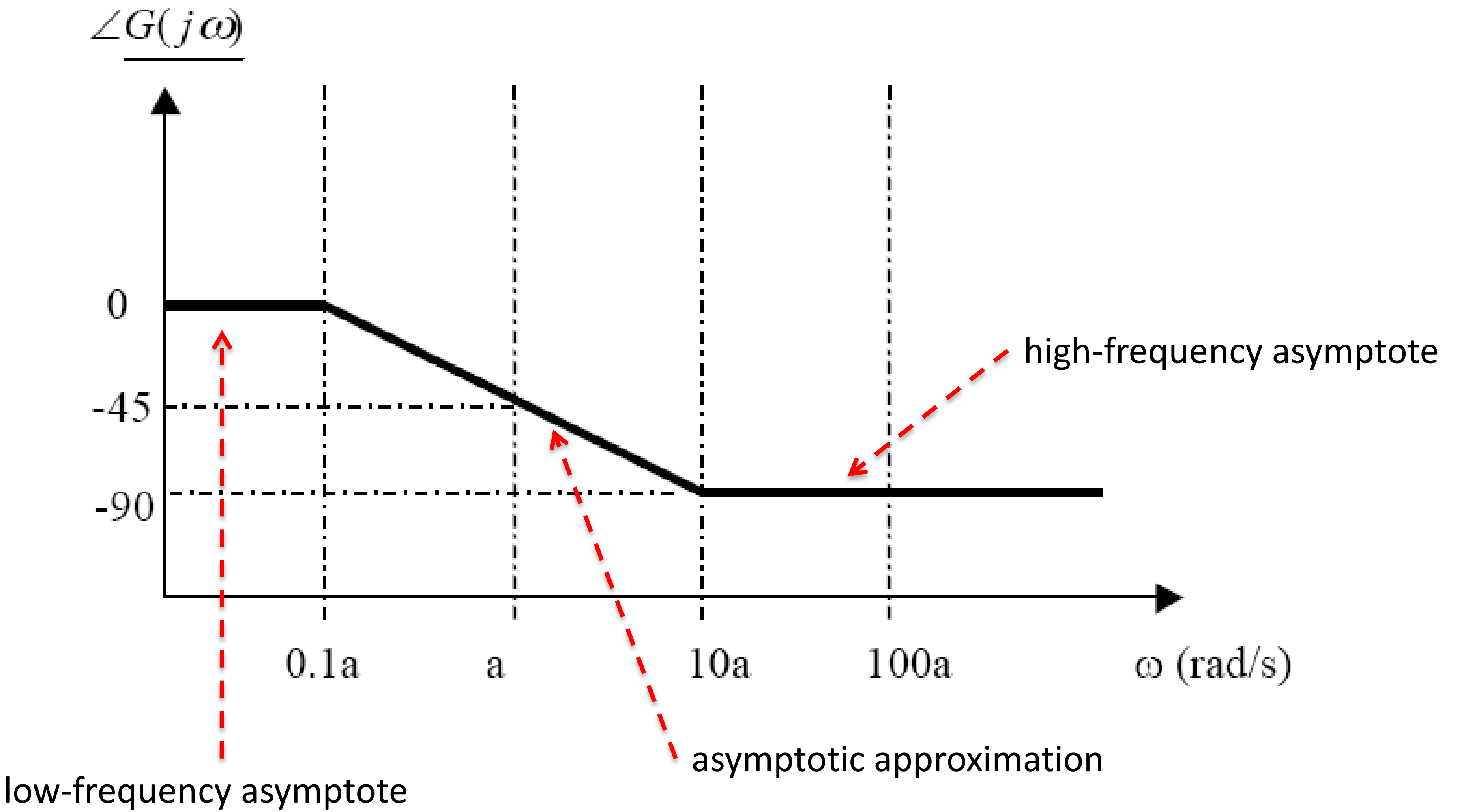
- 20 dB/decade

$$\angle G(j\omega)=-\tan^{-1}\omega=-90^0$$

Bode Plots- Poles



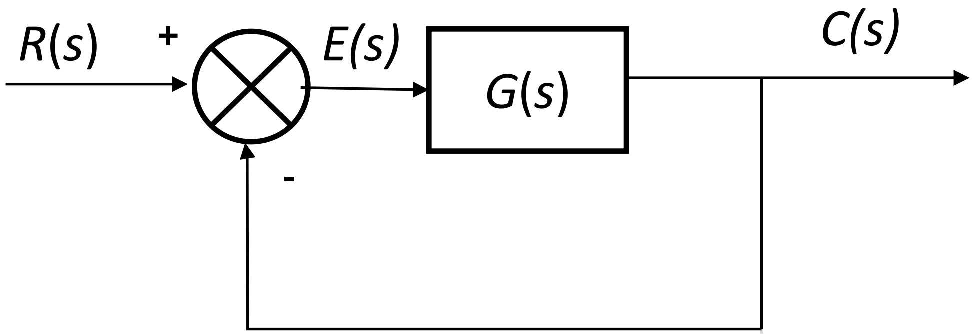
Bode Plots- Poles



Exercise 1

- Sketch the Bode plots for the system shown where

$$G(s) = \frac{K(s+3)}{s(s+1)(s+2)}$$



- Use the command in MATLAB to get the actual bode plot. (use the command 'bode(G)')

Bode Plots: Second Order

- For

$$G(s)=\frac{\omega_n^{\ 2}}{s^2+2\zeta\omega_n s+\omega_n^{\ 2}}$$

$$G(s)=\frac{\omega_n^{\ 2}}{s^2+2\zeta\omega_n s+\omega_n^{\ 2}}=\frac{1}{\frac{s^2}{\omega_n^{\ 2}}+\frac{2\zeta}{\omega_n}s+1}$$

$$G(j\omega)=\frac{1}{\frac{(j\omega)^2}{\omega_n^{\ 2}}+j\frac{2\zeta\omega}{\omega_n}+1}=\frac{1}{\left(1-\frac{\omega^2}{\omega_n^{\ 2}}\right)+j\frac{2\zeta\omega}{\omega_n}}$$

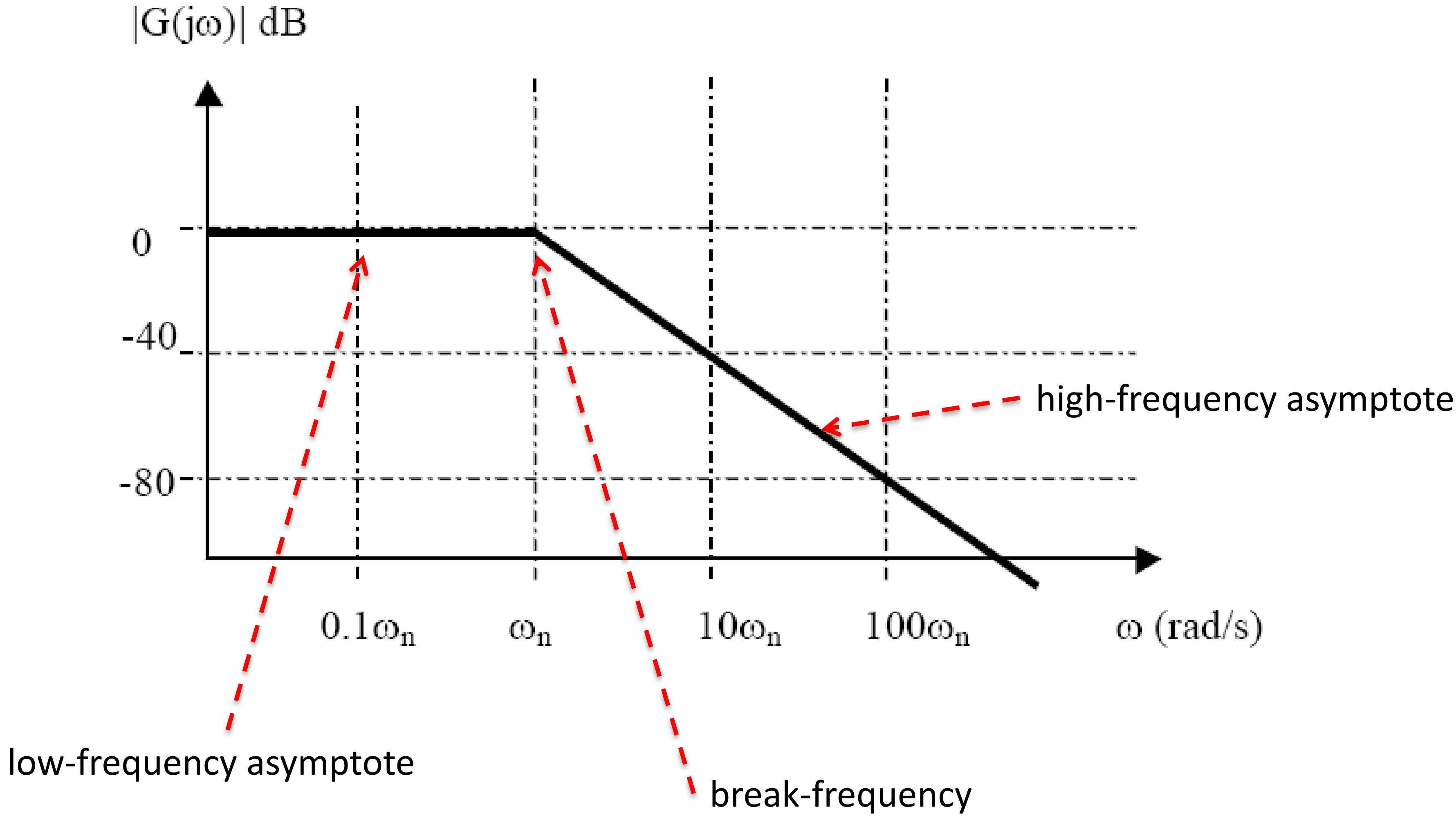
- At low frequency, $\omega << \omega_n$, $G(j\omega)=1$

$$|Gj\omega)|_{dB}=20\log 1=0\,dB; \angle G(j\omega)=0^0$$

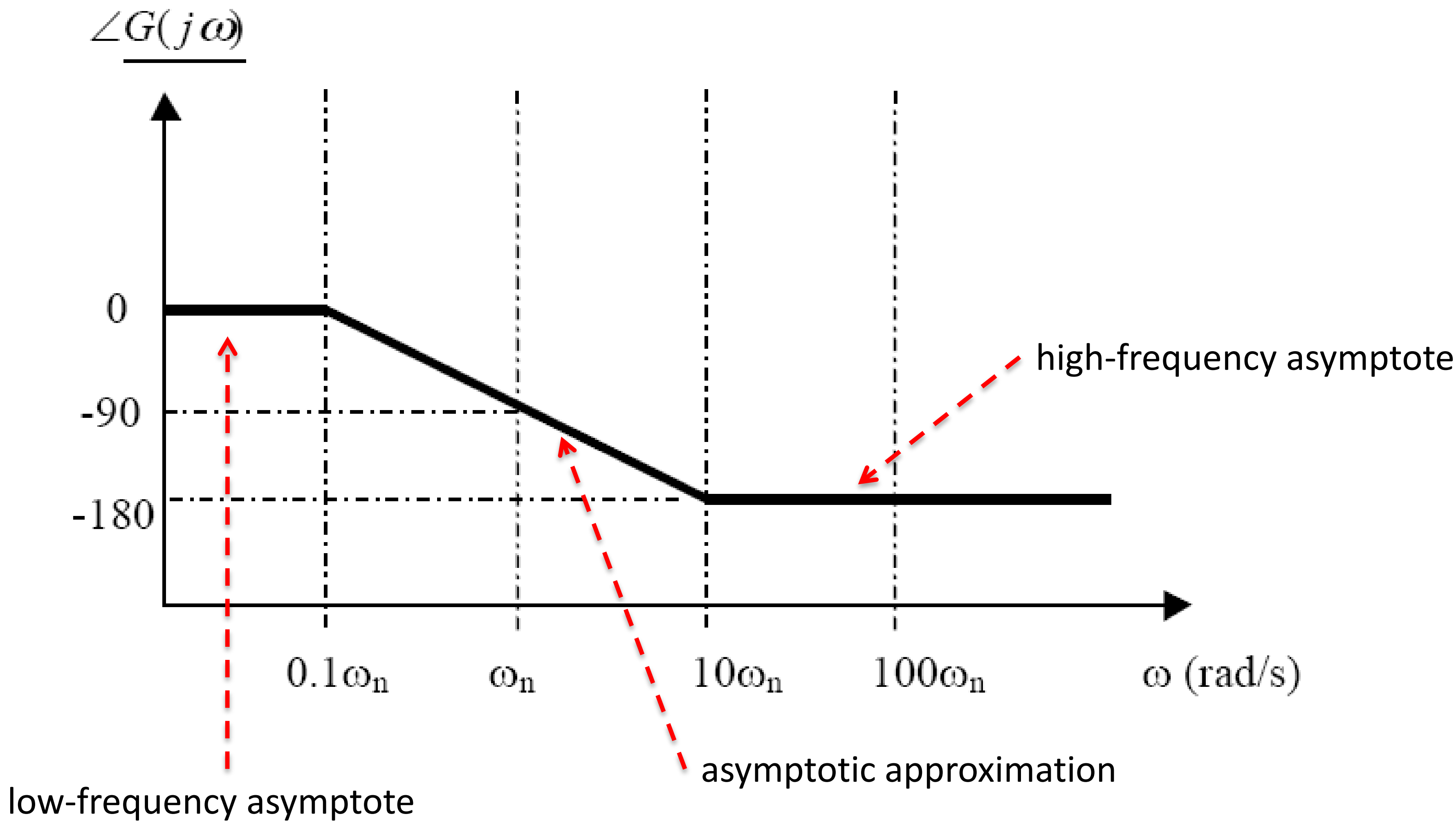
- At high frequency, $\omega >> \omega_n$, $G(j\omega)=1/-\omega^2$

$$|Gj\omega)|_{dB}=-20\log\omega^2=-40\log\omega; \angle G(j\omega)=-180^0$$

Bode Plots: Second Order



Bode Plots: Second Order



Bode Plots: Second Order

$$G(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

normalized $G(s) = \dfrac{1}{\dfrac{s^2}{w_n^2} + \dfrac{2z}{w_n}s + 1}$

$$G(jw) = \left. \frac{1}{\dfrac{s^2}{w_n^2} + \dfrac{2z}{w_n}s + 1} \right|_{s \to jw}$$

At the break frequency, $\omega = \omega_n$

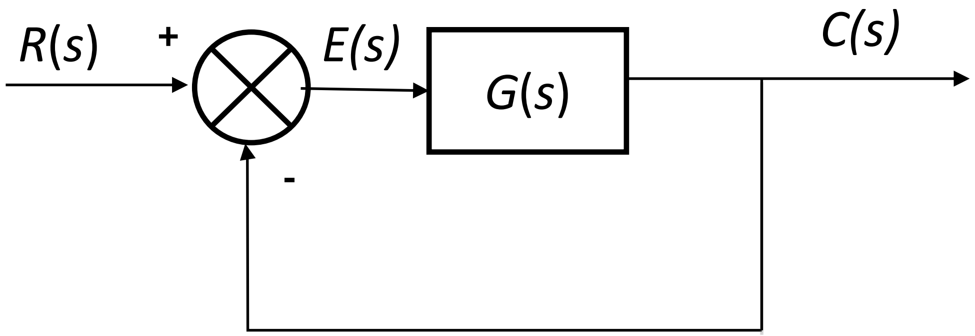
$$G(j\omega) = 1/j2\zeta$$

$$|G(j\omega)|_{dB} = -20\log(2\zeta); \angle G(j\omega) = -90^0$$

Example 2

- Sketch the Bode plot for G(s) for the unity feedback system shown below where

$$G(s) = \frac{(s+3)}{(s+2)(s^2+2s+25)}$$



Obtain the actual bode plot using MATLAB.

5.4 Analysis of Bode Plot

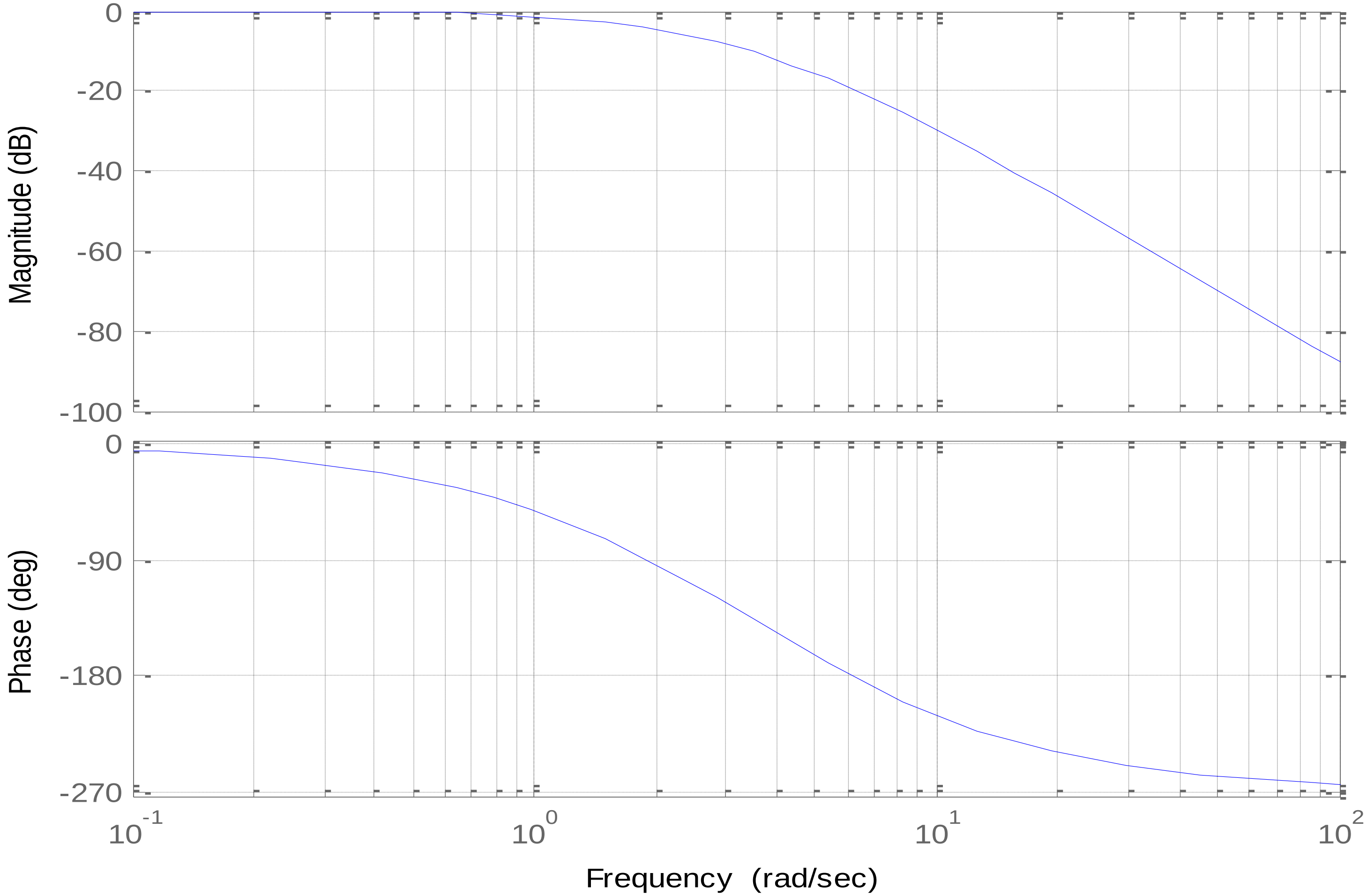
Stability via Bode Plots

- Stability of a closed-loop system can be determined using Bode plot of an open-loop system.
- A closed-loop system is stable if the magnitude of the OL system is less than 0 dB (unity gain) at the frequency where the phase is $\pm 180^0$.
- Example :
- Use Matlab to get the actual plotting
- Determine the range of *K* within which the unity feedback system is stable. Let G(s) = *K*/[(s+2)(s+4)(s+5)].
- Solution: Normalise

$$G(s) = \frac{K}{(40)\left(\frac{s}{2}+1\right)\left(\frac{s}{4}+1\right)\left(\frac{s}{5}+1\right)}$$

- For convenience, choose *K* = 40.

The bode plot when K = 40



Solution 3

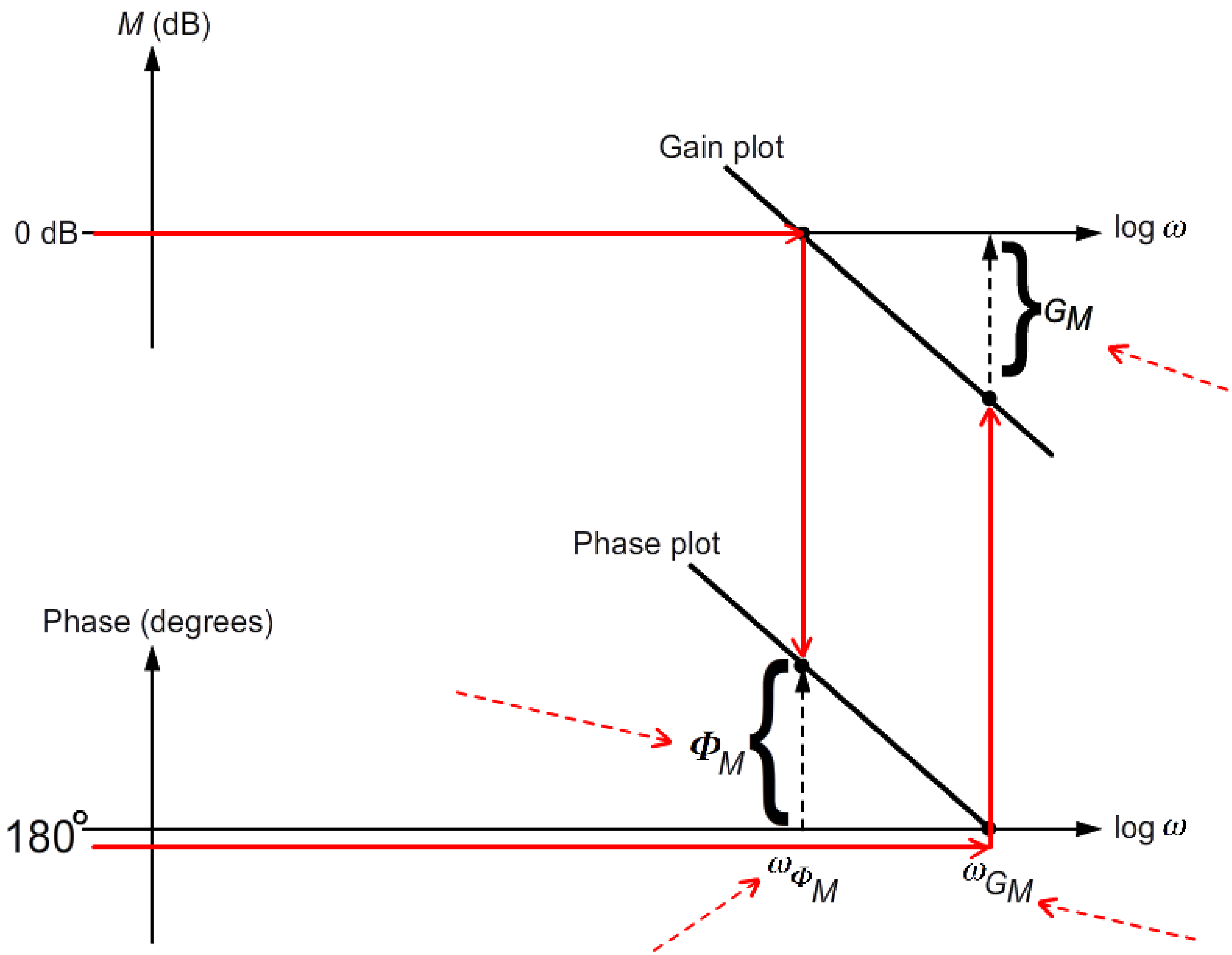
- With K = 40, at phase -180^0, ω = 7 rad/s, magnitude = -20 dB.
- The system is stable at K = 40.
- Therefore, an increase in gain of 20 dB (20 log 10) is possible for stability.
- Hence the gain for stability is 400 (40 x 10).
- Range for stability: $0 < K < 400$.
- Actual results: ω = 6.16 rad/s, K = 378.

Gain and Phase Margins

- Gain margin and phase margin are two quantitative measures of how stable a system is.
- Systems with greater gain and phase margins can withstand greater changes in system parameters before becoming unstable.
- Gain margin, G_M is the change required in open-loop gain at -180^0 of phase shift to make the closed-loop system unstable.
- The gain margin is found by using the phase plot to find the gain margin frequency, ω_{GM} where the phase angle is -180^0.
- At this frequency look at the magnitude plot to determine the gain margin which is the gain required to raise or decrease the magnitude curve to 0 dB.

- Phase margin, Φ_M is the change required in open-loop phase shift to make the closed-loop system unstable.
- The phase margin is found by using the magnitude curve to find the phase margin frequency, $\omega_{\Phi M}$ where the gain is 0 dB.
- At this frequency, the phase margin is the difference between the phase value and -180^0.

Gain and Phase Margins



Example

Consider a unity feedback system :

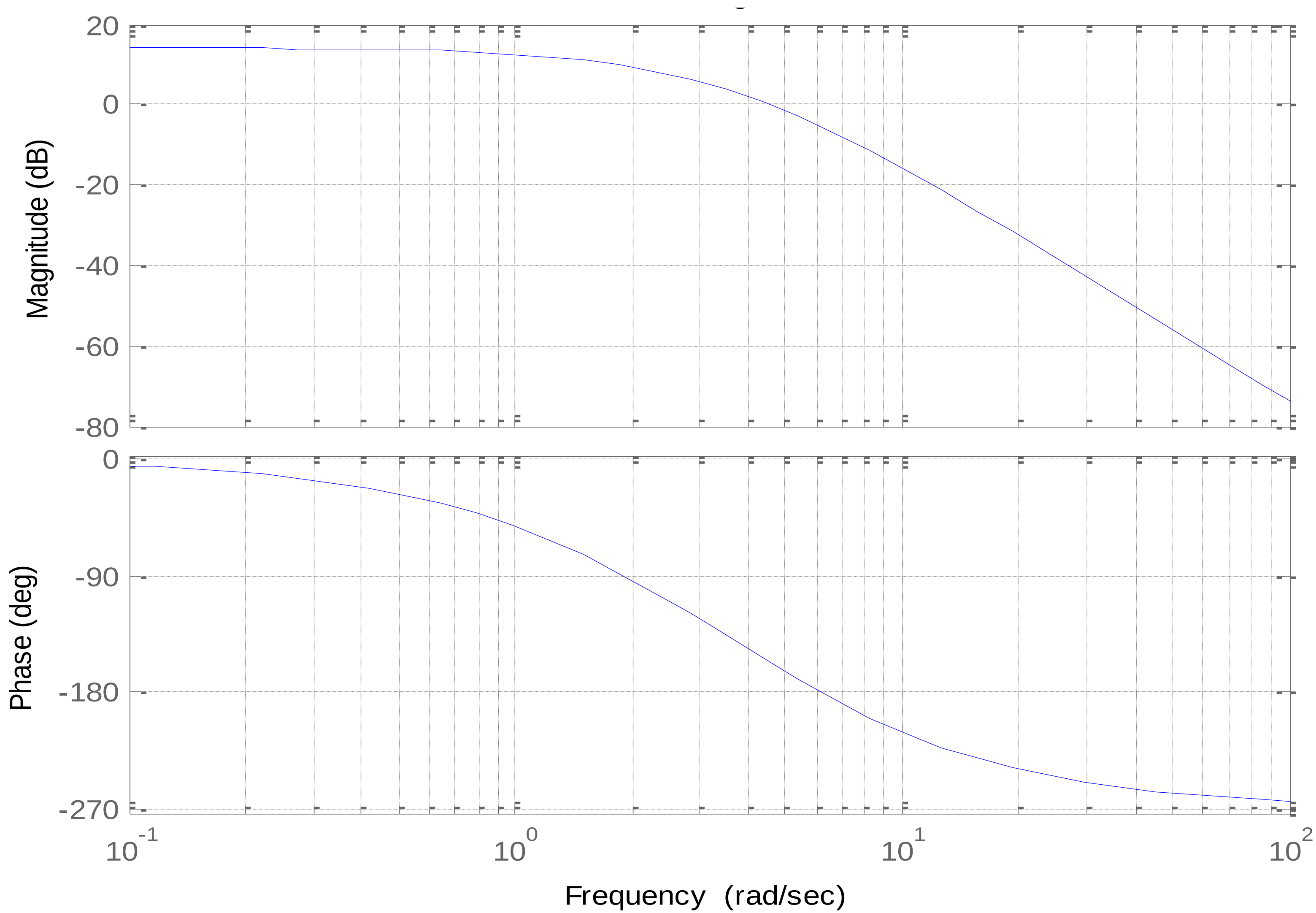
$$G(s) = 200/[(s+2)(s+4)(s+5)]$$

Using Matlab, find the gain margin and the phase margin from the bode plot.

SOLUTION:

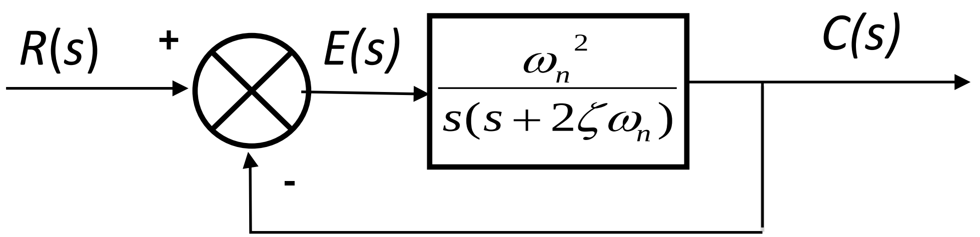
- ω_{GM} = 7 rad/s. G_M = 6.02 dB.
- $\omega_{\Phi M}$ = 5.5 rad/s. Φ_M is = $180^0 - 165^0 = 15^0$.
- Note that: - any additional zeros and/or poles will change the original bode plot. This can be clearly observed by using the command ‘sisotool(G)’ and adding zeros/poles from the menu. Resulted in the changes in G_M and Φ_M, hence the stability

Bode plot of the system :



Relation between Closed-Loop Time and Closed-Loop Frequency Responses

- There is a relationship between closed-loop time and closed-loop frequency responses.
- Consider the second order feedback control system:



$$T(s)=\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$T(j\omega)=\frac{\omega_n^2}{(j\omega)^2+j2\zeta\omega_n\omega+\omega_n^2}=\frac{\omega_n^2}{(\omega_n^2-\omega^2)+j2\zeta\omega_n\omega}$$

$$M=|T(j\omega)|=\frac{\omega_n^2}{\sqrt{(\omega_n^2-\omega^2)^2+4\zeta^2\omega_n^2\omega^2}}$$

- By plotting the closed loop bode plot and looking at the magnitude-log plot, we can measure:

  - The maximum magnitude value of M_p,

$$M_P = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

  - The frequency at the M_p ,

$$\omega_P = \omega_n\sqrt{1-2\zeta^2}$$

- Therefore, we can deduce that the maximum magnitude is directly related to damping ratio and overshoot of a system.

- Bandwidth, ω_{BW} is the frequency at which the magnitude response curve is -3 dB.
- Relationships between the bandwidth and the time response specifications:

$$\omega_{BW} = \omega_n \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2}}$$

$$\omega_{BW} = \frac{4}{T_s \zeta} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2}}$$

$$\omega_{BW} = \frac{\pi}{T_p \sqrt{1-\zeta^2}} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2}}$$

Exercise 3

- Given a unity feedback system with *G(s) = K/[s(s+1)(s+2)].* Using Matlab, find *the* system gain and phase margins, maximum amplitude and bandwidth for *K* = 1 and 10.

ANSWER

Gain and phase margins are obtained from the open-loop Bode plot. [prove using MATLAB]

Maximum amplitude and bandwidth can only be obtained with the closed-loop Bode plot. [prove using MATLAB]

	K = 1	K = 10
Gain margin	29.5 dB	9.5 dB
Phase margin	78^0	25^0
Maximum amplitude	0 dB	7.3 dB
Bandwidth	0.26 rad/s	2 rad/s

Hard to find manually

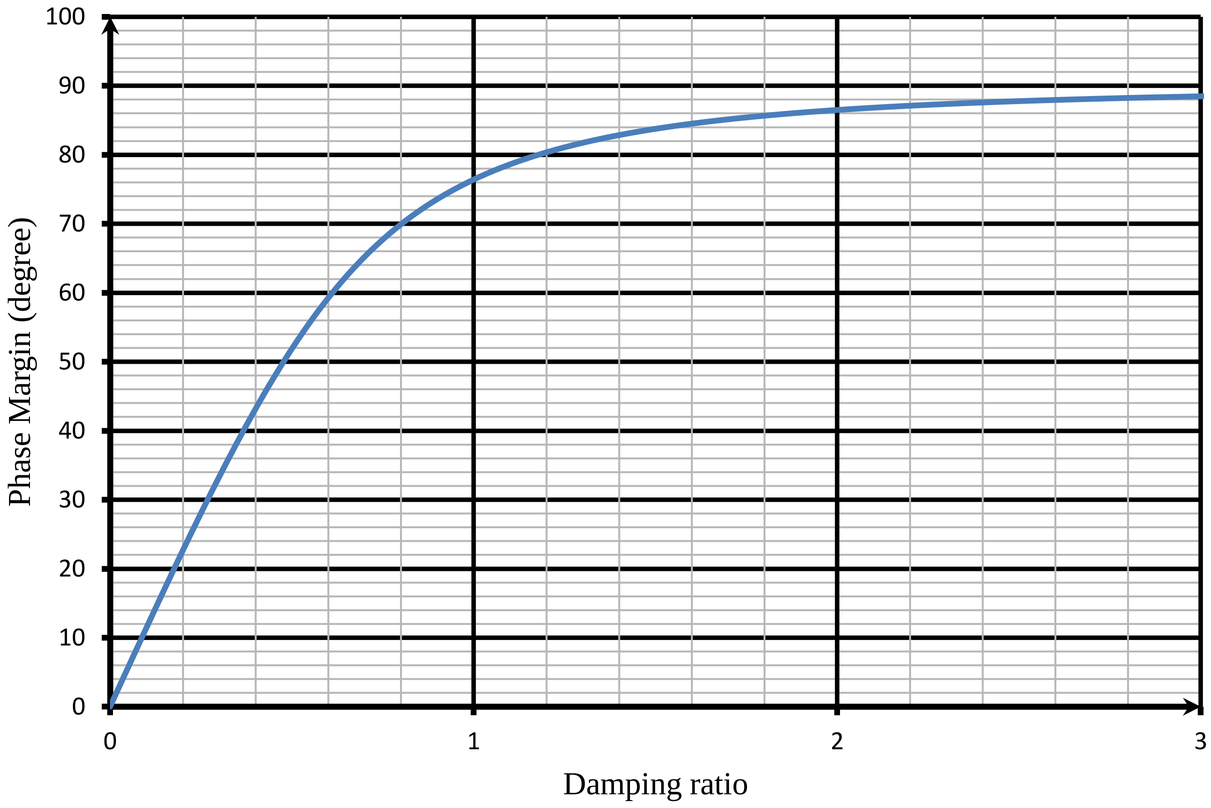
Relation between Closed- and Open-Loop Frequency Responses

- We do not have an easy way of finding the closed-loop frequency response from which we could determine M_p and thus the transient response.
- We can sketch the open-loop frequency response (Bode plot) but not the closed-loop frequency response.
- One of the techniques to obtain the closed-loop frequency response from open-loop frequency response is Nichols Chart which is not covered in this module.

Damping Ratio and Phase Margin

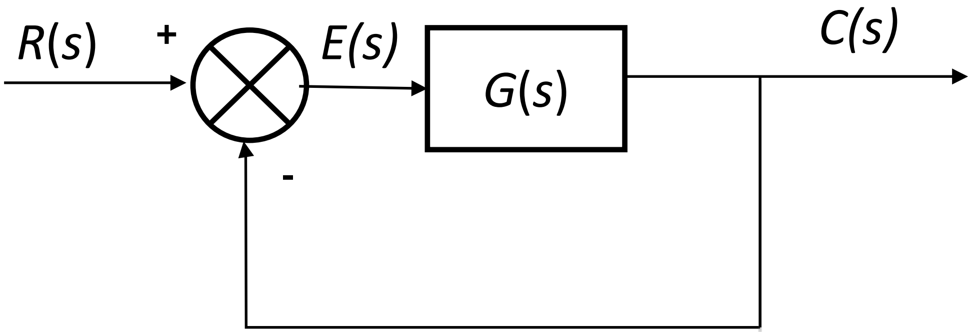
- The relationship between the phase margin and the damping ratio can be derived and given by:

$$\Phi_M = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}$$



Steady-state Error from Frequency Response

- The steady state error can also be found from the open loop bode plot and using the same formula from the time domain analysis
- For a unity feedback system, the steady state error can be further simplified.



$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)}$$

- Hence, the static error constants are related to the input test signal.
- For a unit step input, the steady state error is given by,

$$e_{ss} = \lim_{s \to 0} \frac{s}{1+G(s)} \cdot \frac{1}{s}$$

$$= \frac{1}{1+\lim_{s \to 0} G(s)}$$

$$= \frac{1}{1+K_P}$$

- where, K_P is the position error constant, given by

$$K_P = \lim_{s \to 0} G(s)$$

- For a unit ramp input, the steady state error is given by,

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \cdot \frac{1}{s^2}$$

$$= \lim_{s\to 0} \frac{1}{s + sG(s)}$$

$$= \frac{1}{\lim_{s\to 0} sG(s)}$$

$$= \frac{1}{K_\upsilon}$$

- where, K_v is the velocity error constant, given by

$$K_\upsilon = \lim_{s\to 0} sG(s)$$

- For a parabola input, the steady state error is given by,

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \cdot \frac{1}{s^3}$$

$$= \lim_{s\to 0} \frac{1}{s^2 + s^2 G(s)}$$

$$= \frac{1}{\lim\limits_{s\to 0} s^2 G(s)}$$

$$= \frac{1}{K_a}$$

- where, K_a is the velocity error constant, given by

$$K_a = \lim_{s\to 0} s^2 G(s)$$

Steady-state Error from Frequency Response

- For a unity feedback system with an open-loop transfer function *G(s)*, the steady-state errors can be found identify the system type and using the respective formula:

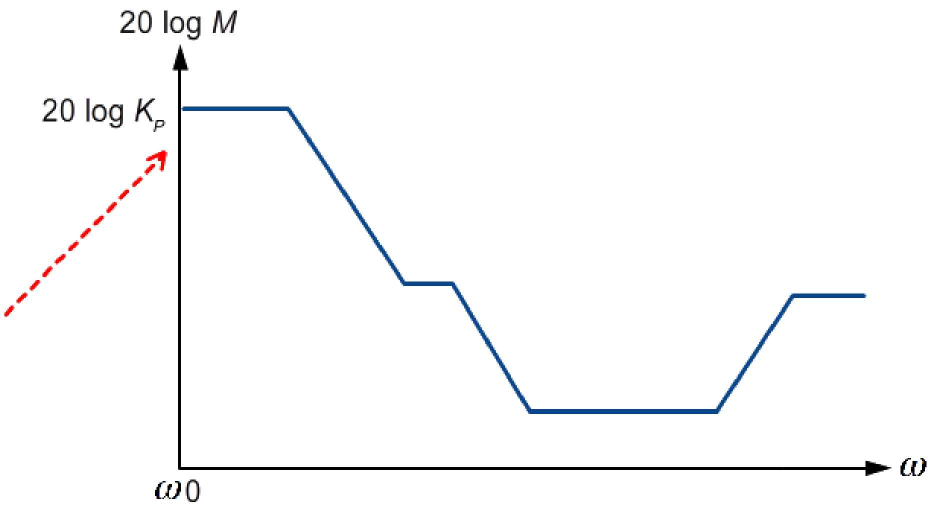
- for system type 0: $$e_{ss} = \frac{1}{1+K_P}$$

- for system type 1: $$e_{ss} = \frac{1}{K_\upsilon}$$

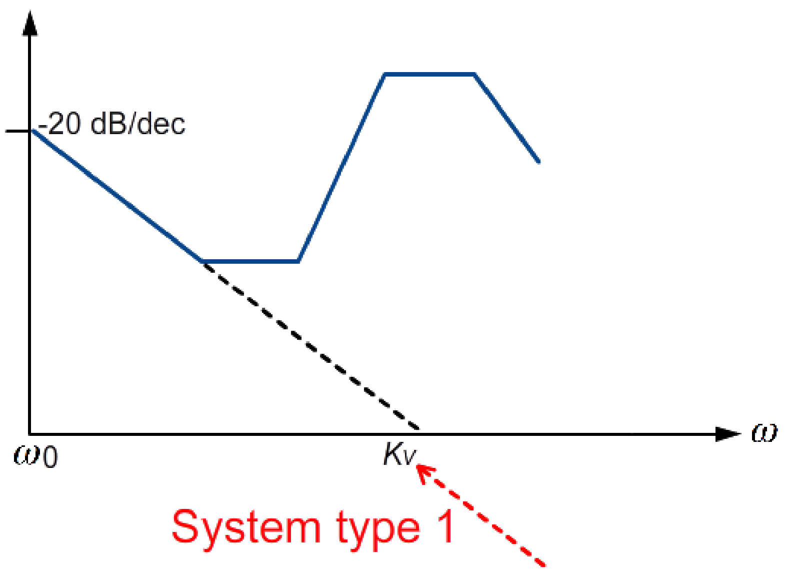
- for system type 2: $$e_{ss} = \frac{1}{K_a}$$

Steady-state Error from Frequency Response

- By identifying the system type from the open-loop Bode plot, the steady state error can be easily found as follows,

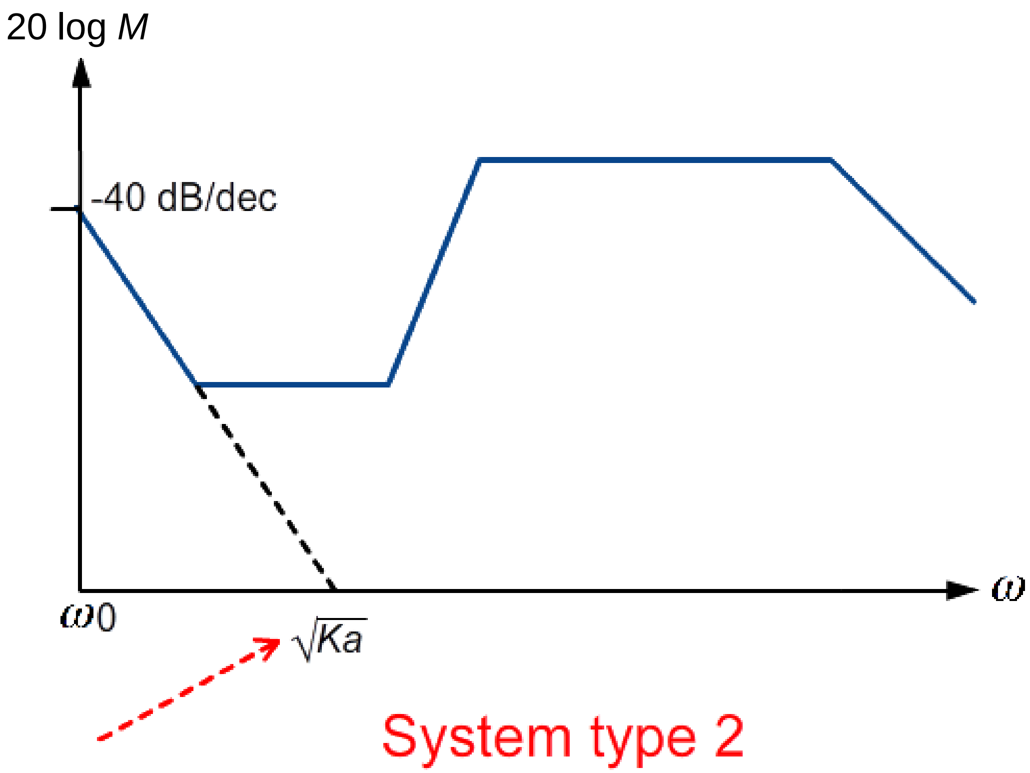


System type 0



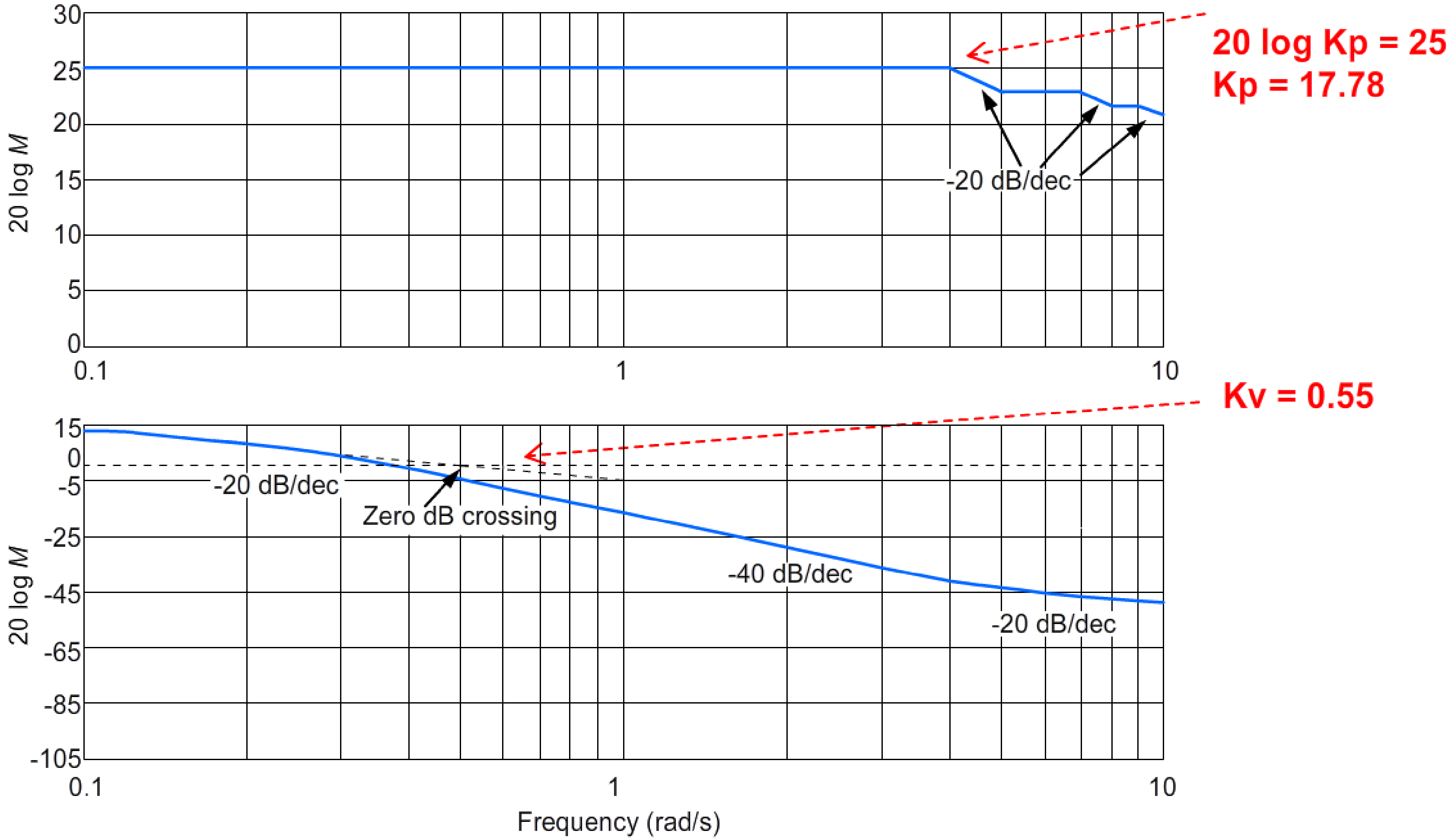
System type 1

Steady-state Error from Frequency Response

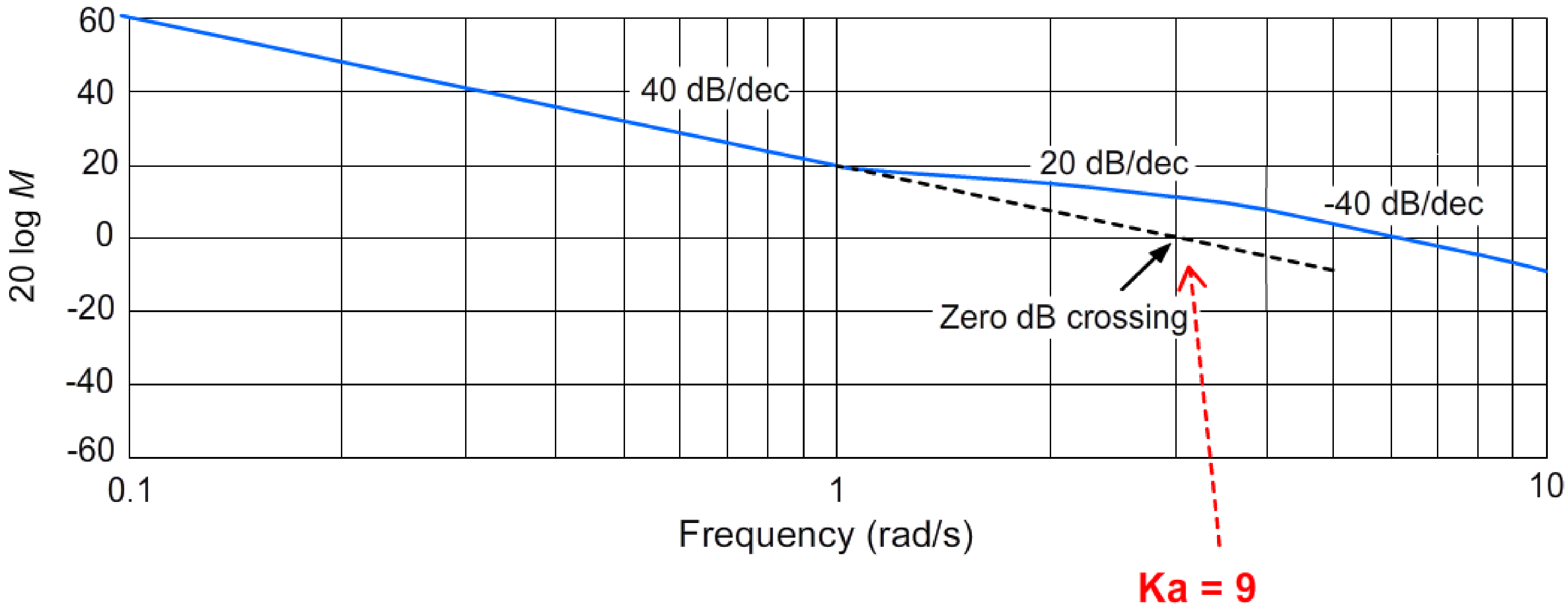


Example

- Find the steady-state error for the following Bode plots:



Example


60
40
20
0
-20
-40
-60
20 log M
40 dB/dec
20 dB/dec
-40 dB/dec
Zero dB crossing
0.1
1
10
Frequency (rad/s)
Ka = 9

Conclusions

We have covered

- ✓ The graphical analysis using Bode Plot
- ✓ The stability analysis by looking at Gain and Phase Margins
- ✓ Some relationships between open loop and closed loop systems' information

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