SCJ2013 Data Structure & Algorithms

Algorithm Efficiency Analysis

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Objectives

At the end of the class, students are expected to be able to do the following:

- Know how to measure algorithm efficiency.
- Know the meaning of big O notation.

Introduction

Algorithm analysis:-

to study the efficiency of algorithms when the input size grow, based on the number of steps, the amount of computer time and space.

Analysis of algorithms

- Is a major field that provides tools for evaluating the efficiency of different solutions

What is an efficient algorithm?

- Faster is better (Time)
  - How do you measure time? Wall clock? Computer clock?
- Less space demanding is better (Space)
  - But if you need to get data out of main memory ittakes time

Analysis of algorithms

- Algorithm analysis should be independent of :
  - Specific implementations and coding tricks (programming language, control statements – Pascal, C, C++, Java)
  - Specific Computers (hw chip, OS, clock speed)
  - Particular set of data (string, int, float)

But size of data should matter

Analysis of algorithms

- Three possible states in algorithm analysis:
  - best case
  - average case
  - worst case

- The worst case is always considered → the maximum boundary for execution time or memory space for any input size.
- Execution time for the worst case → complexity time

Worse Case/Best Case/Average Case

For a particular problem size, we may be interested in:

- **Worst-case efficiency**: Longest running time for *any* input of size *n*
  - A determination of the maximum amount of time that an algorithm requires to solve problems of size *n*
- **Best-case efficiency**: Shortest running time for *any* input of size *n*
  - A determination of the minimum amount of time that an algorithm requires to solve problems of size *n*
- **Average-case efficiency**: Average running time for *all* inputs of size *n*
  - A determination of the average amount of time that an algorithm requires to solve problems of size *n*

Examples of the 3 cases

Algorithm: sequential search of *n* elements

- Best-case: Find the target in the first place the element set. $C(n) = 1$
- Worst-case: Find or cannot find the target after compare every element with the target value. $C(n) = n$
- Average-case: Depends on the probability (p) that the target will be found. C(n) n/2

Big *O* Notation

- Complexity time can be represented by

  Big 'O' notation.

- Big 'O' notation is denoted as

  O(f(n))

  O – "on the order of"

  f(n)- algorithm's growth-rate function that may consist of $1,\ \log_x n,\ n,\ n \log_x n,\ n^2,\ \ldots$

- An algorithm requires time proportional to *f(n)*. *O(f(n)) means* order of *f(n)*.

Big *O* Notation

- Notation that used to show the complexity time of algorithms.

Notation	Execution time / number of step
$O(1)$	Constant function, independent of input size, n Example: Finding the first element of a list.
$O(\log_x n)$	Problem complexity increases slowly as the problem size increases. Squaring the problem size only doubles the time. Charac.: Solve a problem by splitting into constant fractions of the problem (e.g., throw away ½ at each step)
$O(n)$	Problem complexity increases linearly with the size of the input, n Example: counting the elements in a list.

Big *O* Notation

$O(n \log_x n)$	Log-linear increase - Problem complexity increases a little faster than n Characteristic: Divide problem into subproblems that are solved the same way Example: mergesort
$O(n^2)$	Quadratic increase. Problem complexity increases fairly fast, but still manageable Characteristic: Two nested loops of size n
$O(n^3)$	Cubic increase. Practical for small input size, n.
$O(2^n)$	Exponential increase - Increase too rapidly to be practical Problem complexity increases very fast Generally unmanageable for any meaningful n Example: Find all subsets of a set of n elements

Big O Notation

- Example of algorithm (only for **`cout`** operation):

notation	code
O(1) *Constant*	`int counter = 1;` `cout << "Arahan cout kali ke " << counter << "\n";`

O($log_x n$)
Logarithmic
	`int counter = 1; int i = 0;`
	`for (i = x; i <= n; i = i * x) { // x must be > than 1`
	`        cout << "Arahan cout kali ke " << counter << "\n";`
	`        counter++;`
	`}`

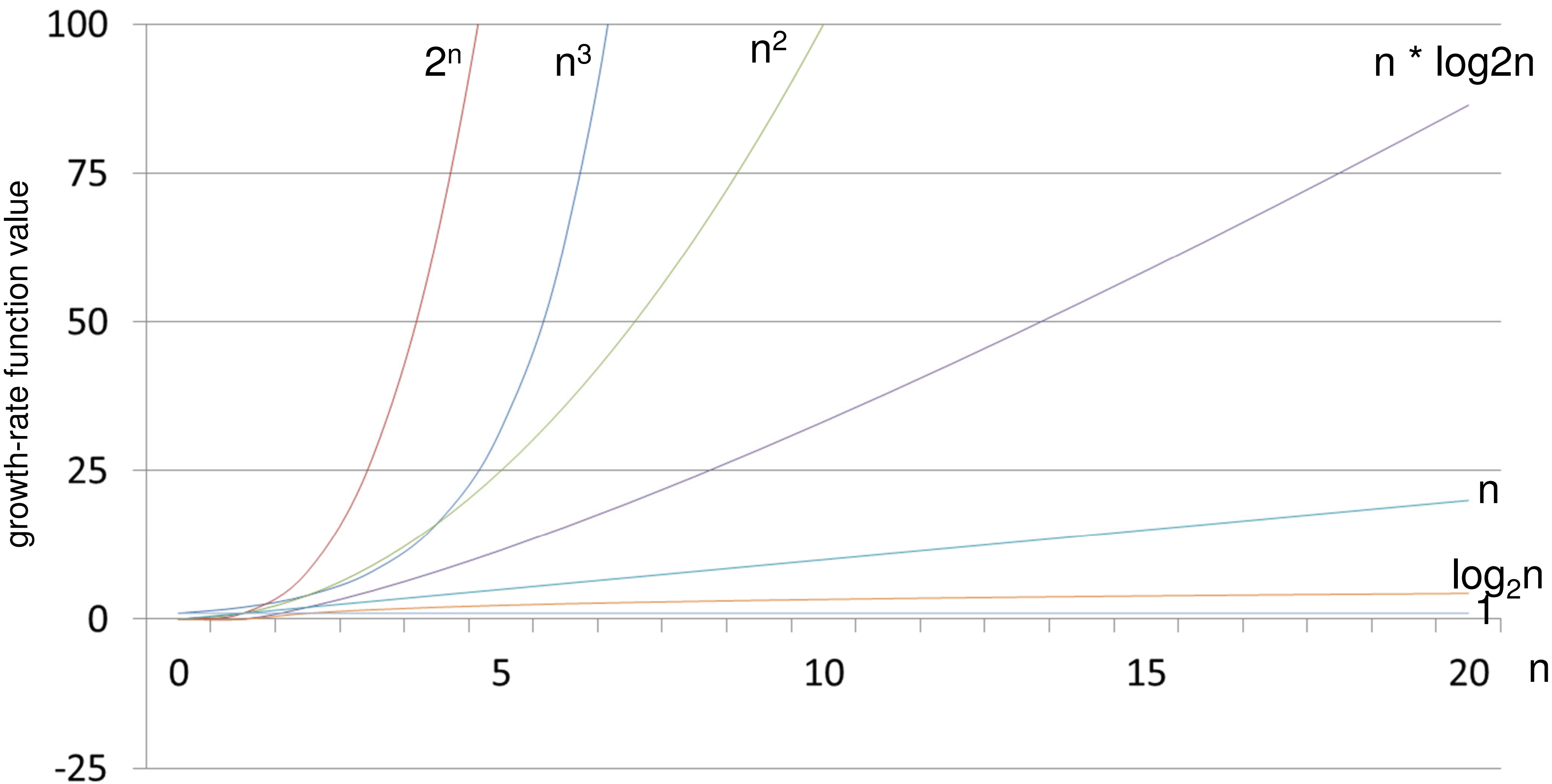
Order of increasing complexity

Order of growth for some common function:

- $O(1) < O(\log_x n) < O(n) < O(n \log_2 n) < O(n^2) < O(n^3) < O(2^n)$

Notasi	**n = 8**	**n = 16**	**n = 32**
$O(\log_2 n)$	3	4	5
$O(n)$	8	16	32
$O(n \log_2 n)$	24	64	160
$O(n^2)$	64	256	1024
$O(n^3)$	512	4096	32768
$O(2^n)$	256	65536	4294967296

Order-of-Magnitude Analysis and Big O Notation



Big *O* Notation

- Example of algorithm for common function:

O(n) *Linear*	`int counter = 1; int i = 0;`
	
	`for (i = 1; i <= n; i++) {`
	`        cout << "Arahan cout kali ke " << counter << "\n";`
	`        counter++;`
	`}`

O(n log$_x$n) *Linear Logarithmic*	`int counter = 1; int i = 0; int j = 1;`
	
	`for (i = x; i <= n; i = i * x) { // x must be > than 1`
	`      while (j <= n) {`
	`        cout << "Arahan cout kali ke " << counter << "\n";`
	`                counter++; j++;`
	`        }`
	`}`

Big *O* Notation

- Example of algorithm for common function:

$O(n^2)$ Quadratic	`int counter = 1;`
	`int i = 0;`
	`int j = 0;`
	
	`for (i = 1; i <= n; i++) {`
	`    for (j = 1; j <= n; j++) {`
	`        cout << "Arahan cout kali ke " <<`
	`counter << "\n";`
	`                counter++;`
	`        }`
	`}`

Big *O* Notation

- Example of algorithm for common function:

$O(n^3)$ Cubic	`int counter = 1;`
	`int i = 0;`
	`int j = 0;`
	`int k = 0;`
	
	`for (i = 1; i <= n; i++) {`
	`      for (j = 1; j <= n; j++) {`
	`            for (j = 1; j <= n; j++) {`
	`                  cout << "Arahan cout kali ke " <<`
	`counter << "\n";`
	`                              counter++;`
	`               }`
	`        }`
	`}`

Big *O* Notation

- Example of algorithm for common function:

$O(2^n)$ Exponential	`int counter = 1;`
	`int i = 1;`
	`int j = 1;`
	
	`while (i <= n) {`
	`        j = j * 2;`
	`        i++;`
	`}`
	
	`for (i = 1; i <= j; i++) {`
	`        cout << "Arahan cout kali ke " << counter << "\n";`
	`        counter++;`
	`}`

Determine the complexity time of algorithm

Can be determined

- theoretically – by calculation
- practically – by experiment or implementation

Determine the complexity time of algorithm - practically

- Implement the algorithms in any programming language and run the programs
- Depend on the compiler, computer, data input and programming style.

Determine the complexity time of algorithm - theoretically

- The complexity time is related to the number of steps /operations.
- Complexity time can be determined by

  1. Count the number of steps and then find the class of complexity.

  Or

  2. Find the complexity time for each steps and then count the total.

Determine the number of steps

- The following algorithm is categorized as O(n).

```
int counter = 1;

int i = 0;

for (i = 1; i <= n; i++) {

   cout << "Arahan cout kali ke “;

   cout << counter << "\n";

   counter++;

}
```

Determine the number of steps

Num	statements
1	int counter = 1;
2	int i = 0;
3	i = 1
4	i <= n
5	i++
6	cout << "Arahan cout kali ke " << counter << "\n"
7	counter++

Determine the number of steps

- Statement 3, 4 & 5 are the loop control and can be assumed as one statement.

Num	Statements
1	int counter = 1;
2	int i = 0;
3	i = 1; i <= n; i++
6	cout << "Arahan cout kali ke " << counter << "\n"
7	counter++

Determine the number of steps- summation series

- statement 3, 6 & 7 are in the repetition structure.
- It can be expressed by summation series

$$\sum_{i=1}^{n} f(i) = f(1) + f(2) + \ldots + f(n) = n$$

Where

$f(i)$ – statement executed in the loop

Determine the number of steps-summation series

- example:- if n = 5, i = 1

$$\sum_{i=1}^{5} f(i) = f(1) + f(2) + f(3) + f(4) + f(5) = 5$$

The statement that represented by $f(i)$ will be repeated 5 times

Determine the number of steps-summation series

- example:- if n = 5, i = 3

$$\sum_{i=3}^{5} f(i) = f(3) + f(4) + f(5) = 3$$

The statement that represented by *f(i)* will be repeated 3 time

Determine the number of steps- summation series

- Example: if n = 1, i = 1

$$\sum_{i=1}^{1} f(i) = f(1) = 1$$

The statement that represented by *f(i)* will be executed only once.

Determine the number of steps

statements	Number of steps
int counter = 1;	$\sum_{i=1}^{1} f(i) = 1$
int i = 0;	$\sum_{i=1}^{1} f(i) = 1$
i = 1; i= n; i++	$\sum_{i=1}^{n} f(i) = n$
cout << "Arahan cout kali ke " << counter << "\n"	$\sum_{i=1}^{n} f(i) . \sum_{i=1}^{1} f(i) = n . 1 = n$
counter++	$\sum_{i=1}^{n} f(i) . \sum_{i=1}^{1} f(i) = n . 1 = n$

Determine the number of steps

- Total steps:

  $1 + 1 + n + n + n = 2 + 3n$

  Consider the largest factor.
- Algorithm complexity can be categorized as $O(n)$

Determine the number of steps

Algorithm	Number of Steps
void sample4 ( )	0
{	0
for (int a=2; a<=n; a++)	n-2+1=n-1
cout << " Contoh kira langkah ";	(n-1).1=n-1
}	0
Total Steps	**2(n-1)**

- Total steps =2(*n-1*), *Complexity Time = O (n)*

Algorithm	Number of steps
void sample5 ( )	0
{	0
for (int a=1; a<=n-1; a++)	n-1-1+1=n-1
cout << " Contoh kira langkah ";	(n-1).1=n-1
}	0
Total steps	**2(n-1)**

Determine the number of steps

Continued......

Algorithm	Number of Steps
void sample7 ( )	0
{	0
for (int a=1; a<=n; a++)	n-1+1=n
for (int b=1; b<=a; b++)	n.(n+1)/2
cout << " Contoh kira langkah ";	n.(n+1)/2
}	0
Total steps	**$2n+n^2$**

- Total steps $=2n+n^2$

  Complexity Time= $O\ (n^2)$

Determine the number of steps

Continued...

$$\sum_{a=1}^{n} \sum_{b=1}^{n} 1 =$$

$$= n(1 + 2 + 3 + 4 + \ldots + n)$$

$$= \frac{n(n+1)}{2}$$

$$= \frac{n^2 + n}{2}$$

- To get n.(n+1)/2,
  we used summation series as shown above:

Determine the number of steps - exercise

Count the number of steps and find the Big ‘O’ notation for the following algorithm

```
int counter = 1;

int i = 0;

int j = 1;


for (i = 3; i <= n; i = i * 3) {
   while (j <= n) {
     cout << "Arahan cout kali ke " << counter << "\n";
     counter++;
     j++;
     }
}
```

Determine the number of steps - solution

statements	Number of steps
int counter = 1;	$\sum_{i=1}^{1} f(i) = 1$
int i = 0;	$\sum_{i=1}^{1} f(i) = 1$
int j = 1;	$\sum_{i=1}^{1} f(i) = 1$
i = 3; i <= n; i = i * 3	$\sum_{i=3}^{n} f(i) = f(3) + f(9) + f(27) + \ldots + f(n) = \log_3 n$
j <= n	$\sum_{i=3}^{n} f(i) \;.\; \sum_{j=1}^{n} f(i) = \log_3 n \;.\; n$

Determine the number of steps - solution

`cout << "Arahan cout kali ke " << counter << "\n";`	$\sum_{i=3}^{n} f(i) \ . \ \sum_{j=1}^{n} f(i) \ . \ \sum_{i=1}^{1} f(i) = \log_3 n \ . \ n \ . \ 1$
`counter++;`	$\sum_{i=3}^{n} f(i) \ . \ \sum_{j=1}^{n} f(i) \ . \ \sum_{i=1}^{1} f(i) = \log_3 n \ . \ n \ . \ 1$
`j++;`	$\sum_{i=3}^{n} f(i) \ . \ \sum_{j=1}^{n} f(i) \ . \ \sum_{i=1}^{1} f(i) = \log_3 n \ . \ n \ . \ 1$

Determine the number of steps - solution

Total steps:

$\Rightarrow 1 + 1 + 1 + \log_3 n + \log_3 n \cdot n + \log_3 n \cdot n \cdot 1 + \log_3 n \cdot n \cdot 1 + \log_3 n \cdot n \cdot 1$

$\Rightarrow 3 + \log_3 n + \log_3 n \cdot n + \log_3 n \cdot n + \log_3 n \cdot n + \log_3 n \cdot n$

$\Rightarrow 3 + \log_3 n + 4n \log_3 n$

Determine the number of steps : solution

$$3 + \log_3 n + 4n \log_3 n$$

- Consider the largest factor

  ($4n \log_3 n$)

- and remove the coefficient

  ($n \log_3 n$)

- In asymptotic classification, the base of the log can be omitted as shown in this formula:

  $\log_a n = \log_b n / \log_b a$

- Thus, $\log_3 n = \log_2 n / \log_2 3 = \log_2 n / 1.58...$
- Remove the coefficient 1/1.58..
- So we get the complexity time of the algorithm is

  $O(n \log_2 n)$

Determine the number of steps

Algorithm	No. of Steps
void sample8 ( )	0
{	0
int n, x, i=1;	1
while (i<=n)	n
{	0
x++;	n.1 = n
i++;	n.1 = n
}	0
}	0
Number of Steps	**1 + 3n**

Consider the largest factor : 3n

and remove the coefficient : O(n)

Conclusion and Summary

- Algorithm analysis to study the efficiency of algorithms when the input size grow, based on the number of steps, the amount of computer time and space
- Can be done using Big O notation by using growth of function.
- Order of growth for some common function:

$$O(1) < O(\log_x n) < O(n) < O(n \log_2 n) < O(n^2) < O(n^3) < O(2^n)$$

- Three possible states in algorithm analysis best case, average case and worst case.

References

1. Frank M. Carano, Janet J Prichard. "*Data Abstraction and problem solving with C++*" *Walls and Mirrors*. 5th edition (2007). Addision Wesley.
2. Nor Bahiah et al. *Struktur data & algoritma menggunakan C++. Penerbit UTM, 2005.*