SEE 2523 Theory Electromagnetic

Chapter 2 Electric Fields

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Electromagnetic Fields (Maxwell's Equations)

1. Modern electromagnetism is based on four fundamental relations

Gauss's Law

$$\vec{\nabla} \cdot \vec{D} = \rho_v$$

Faraday's Law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Electric Phenomenon

Gauss's Law

$$\vec{\nabla} \cdot \vec{B} = 0$$

Ampere's Law

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Magnetic Phenomenon

where $\vec{E}$ **is the electric field,** $\vec{H}$ **is the magnetic field,**

$\vec{D}$ **is the electric flux density or electric displacement,**

$\vec{B}$ **is the magnetic flux density,** $\vec{J}$ **is the current density,**

ρv **is the charge density.**

Electrostatic Fields

1. In the static case, all charges are permanently fixed in space.

2. If the charges move, they move at steady rate, so ρ_v and $\vec{J}$ are constant in time ($d\vec{B}/dt = 0$)

3. Thus, for electrostatics, Maxwell's equations are:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_v}{\varepsilon_o} \quad \text{(a)}$$

$$\vec{\nabla} \times \vec{E} = 0 \quad \text{(b)}$$

(a) The electric field intensity over any closed surface in free space is equal to the total charge enclosed in the surface.

(b) The static electric fields are irrotational.

Coulomb's law (Experimental law) (1)

1. Coulomb's law states that the force *F* between two point charges Q_1 and Q_2 with distance *R* is:

   a) Directly proportional to the product Q_1Q_2 of the charges.

$$F \propto Q_1Q_2$$

   b) Inversely proportional to the square of the distance *R* between them.

$$F \propto \frac{1}{R^2}$$

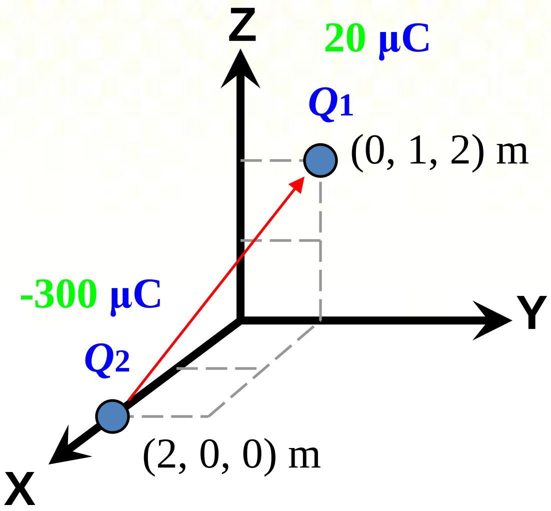
2. Formulation:

$$\vec{F} = \frac{kQ_1Q_2}{R^2}\hat{a}_n$$

where *k* is the proportionality constant depends on the choice of system.

Coulomb's law (2)

Example



Determine force *F* between two point charges *Q*1 and *Q*2 with distance *R*21

Step 1

$$\vec{R}_{21} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz = (0-2)\hat{x} + (1-0)\hat{y} + (2-0)\hat{z} = -2\hat{x} + \hat{y} + 2\hat{z}$$

$$\hat{a}_{21} = \frac{\vec{R}_{21}}{|R_{21}|} = \frac{-2\hat{x} + \hat{y} + 2\hat{z}}{3}$$

Step 2

$$\vec{F} = \frac{kQ_1Q_2}{R_{21}^2}\hat{a}_{21} = \frac{kQ_1Q_2}{(3\,\mathrm{m})^2}\left(\frac{-2\hat{x} + \hat{y} + 2\hat{z}}{3}\right) \quad \mathrm{N}$$

$$k = \frac{1}{4\pi\varepsilon_o}$$

Step 3

$$\vec{F} = \frac{(20\times10^{-6}\,\mathrm{C})(-300\times10^{-6}\,\mathrm{C})}{4\pi\varepsilon_o(3\,\mathrm{m})^2}\left(\frac{-2\hat{x} + \hat{y} + 2\hat{z}}{3}\right) = 6\left(\frac{2\hat{x} - \hat{y} - 2\hat{z}}{3}\right)\,\mathrm{N} = 4\hat{x} - 2\hat{y} - 4\hat{z} \quad \mathrm{N}$$

Gauss's law (Experimental law)

1. Electric field intensity, $\vec{E}$ is the force per unit charge when placed in an electric field.

$$\vec{E} = \frac{\vec{F}}{Q} = \frac{kQ}{R^2}\hat{a}_n = \frac{Q}{4\pi\varepsilon_o R^2}\hat{a}_n$$

2. Gauss's law state that the electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

$$Q = \oint_S \vec{D} \cdot d\vec{S}$$

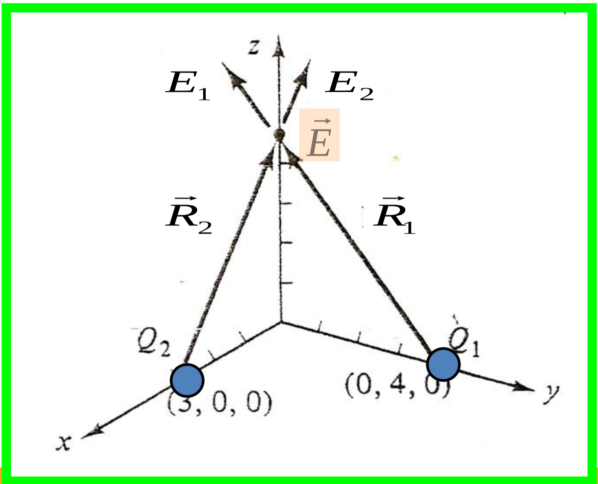
Electric Intensity due to Multiple Point Charges (1)

1. If more than one charge at a different location in a vacuum, the total electric field, $\vec{E}$ in the space external to the location of these charges is the **vector summation of the electric field originating from each individual charge**.

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots + \vec{E}_N$$
$$= \sum_{n=1}^{N} \vec{E}_n$$

Example:

There has a point charge $Q_1 = 0.35\,\mu C$ at (0, 4, 0) m and another point charge $Q_2 = -0.55\,\mu C$ at (3, 0, 0) m. Determine the total electric intensity, $\vec{E}$ at (0, 0, 5) m due to the both charges.



$$\vec{R}_1 = -4\hat{y} + 5\hat{z}$$

$$\vec{R}_2 = -3\hat{x} + 5\hat{z}$$

$$\vec{E}_1 = \frac{0.35\times10^{-6}}{4\pi\varepsilon_o(41)}\left(\frac{-4\hat{y}+5\hat{z}}{\sqrt{41}}\right)\ \text{Vm}^{-1}$$

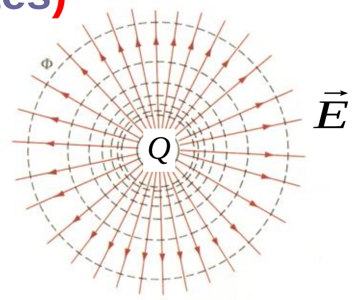
$$\vec{E}_2 = \frac{-0.55\times10^{-6}}{4\pi\varepsilon_o(34)}\left(\frac{-3\hat{y}+5\hat{z}}{\sqrt{34}}\right)\ \text{Vm}^{-1}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$
$$= 74.9\,\hat{x} - 48.0\,\hat{y} - 64.9\,\hat{z}$$

Distribution of Charges (1)

1) Electric field due to point charges (Spherical Coordinates)

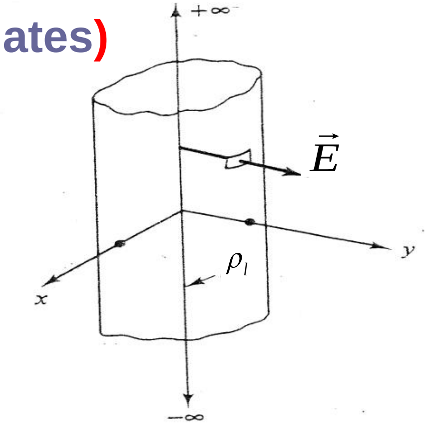
$$\vec{E} = \frac{Q}{4\pi\varepsilon_o r^2}\hat{a}_r$$



2) Electric field due to line charges (Cylindrical Coordinates)

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon_o r}\hat{a}_r$$

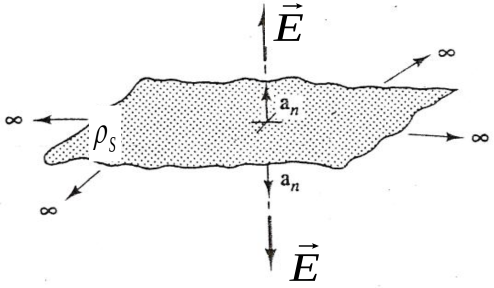
ρ_l **is the line charge density** $(\mathrm{C/m})$



3) Electric field due to surface charges (Cylindrical Coordinates)

$$\vec{E} = \frac{\rho_s}{2\varepsilon_o}\hat{a}_n$$

ρ_S **is the surface charge density** $(\mathrm{C/m^2})$



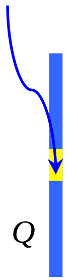
Distribution of Charges (2)

4) To determine the charge, *Q* for each distributions:

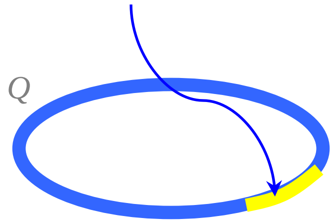
Line charge

$$dQ = \rho_l dl$$
$$Q = \int_l \rho_l dl$$

$$dQ = \rho_l \, dz$$



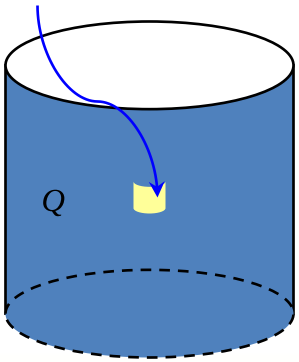
$$dQ = \rho_l \, \rho \, d\phi$$



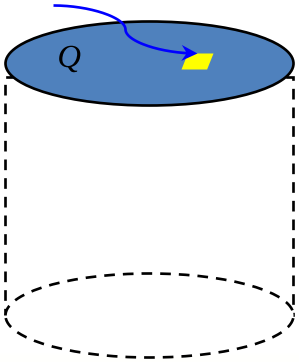
Surface charge

$$dQ = \rho_s dS$$
$$Q = \int_s \rho_s dS$$

$$dQ = \rho_s \, \rho \, d\phi \, dz$$



$$dQ = \rho_s \, \rho \, d\rho \, d\phi$$



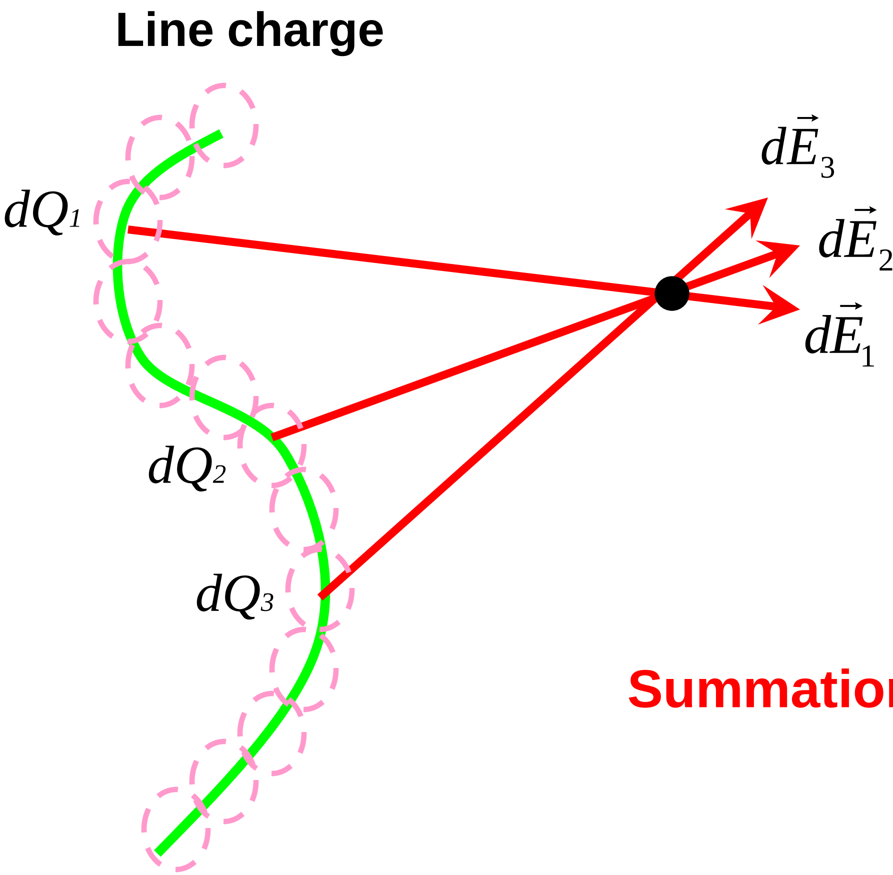
Volume charge (Special cases)

$$dQ = \rho_v dv$$
$$Q = \int_v \rho_v dv$$

Distribution of Charges (3)

Electric Field of a Line Charge

$$\vec{E} = d\vec{E}_1 + d\vec{E}_2 + \ldots + d\vec{E}_N$$



$$\vec{E} = \frac{1}{4\pi\varepsilon_o R^2}\left(dQ_1\,\hat{a}_1 + dQ_2\,\hat{a}_2 + \ldots + dQ_N\,\hat{a}_N\right)$$

$$= \frac{1}{4\pi\varepsilon_o R^2}\sum_{n=1}^{N} dQ_n\hat{a}_n$$

$$\approx \frac{1}{4\pi\varepsilon_o R^2}\int dQ\,\hat{a}_r$$

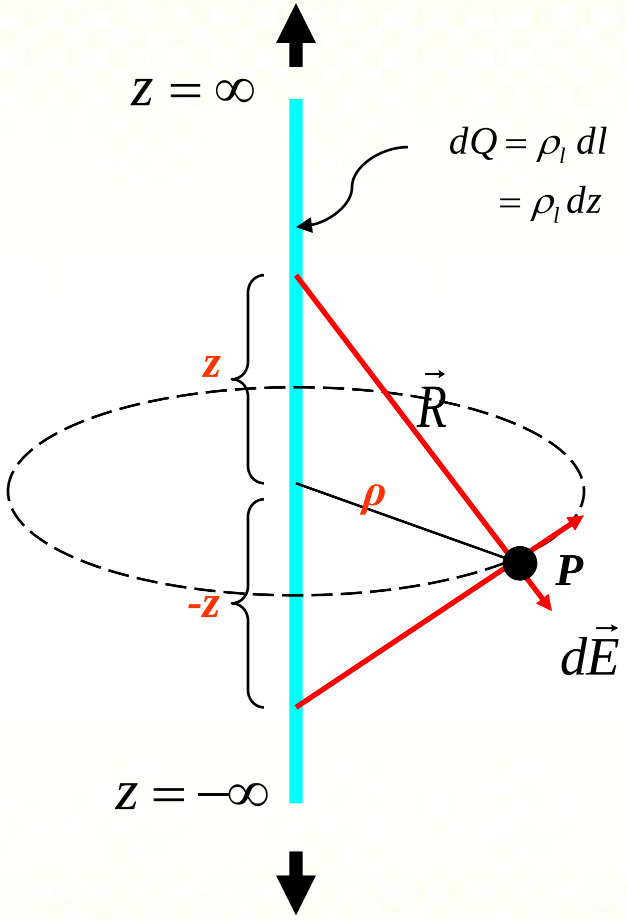
$$dQ_1 = dQ_2 = \ldots = dQ_N$$

Summation = Integration, if $dQ \rightarrow 0$

$$\sum \Rightarrow \int$$

Distribution of Charges (4)

Electric Field of a Line Charge



$$d\vec{E} = \frac{dQ}{4\pi\varepsilon_o R^2}\left(\frac{\rho\hat{\rho} + (z - z')\hat{z}}{\sqrt{\rho^2 + z^2}}\right)$$

$$= \frac{\rho_l\, dz}{4\pi\varepsilon_o\left(\sqrt{\rho^2 + z^2}\right)^2}\left(\frac{\rho\hat{\rho} + (z - z')\hat{z}}{\sqrt{\rho^2 + z^2}}\right)$$

The component z is cancel out, the charge is contribute from location z and –z.

$$\vec{E} = \int_{-\infty}^{\infty}\frac{\rho_l\, dz}{4\pi\varepsilon_o\left(\sqrt{\rho^2 + z^2}\right)^2}\left(\frac{\rho\hat{\rho}}{\sqrt{\rho^2 + z^2}}\right)$$

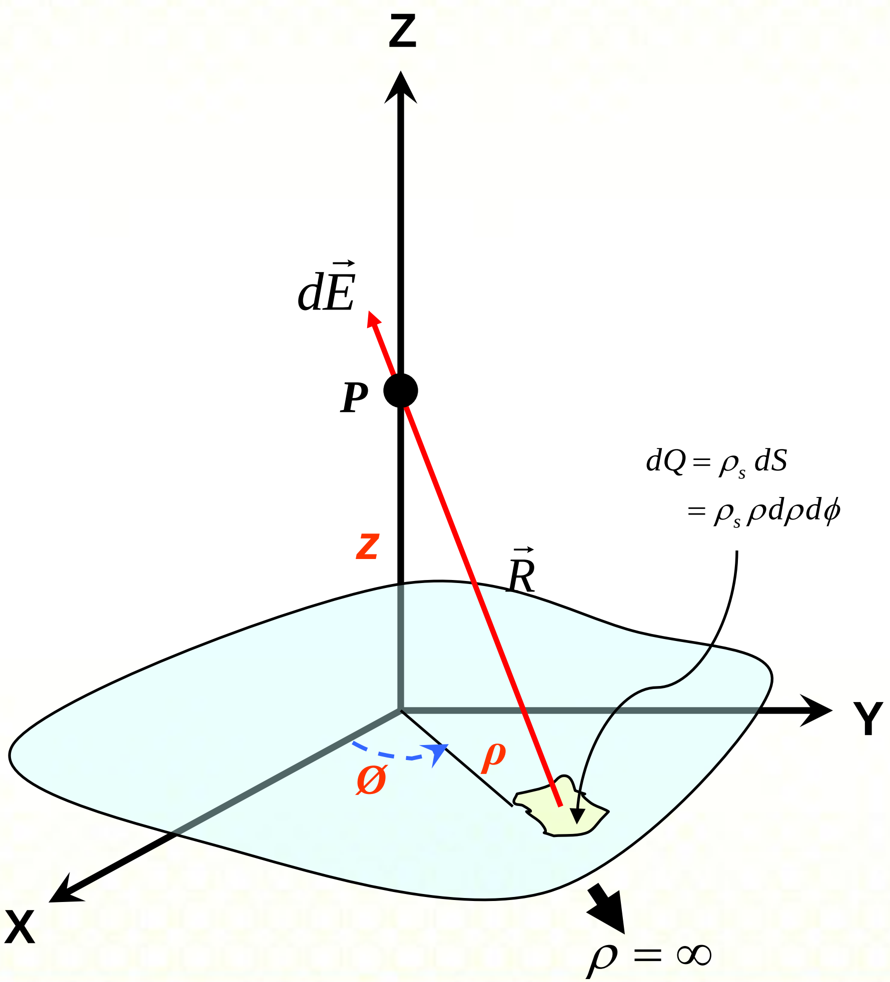
$$= \int_{-\infty}^{\infty}\frac{\rho_l\, \rho\, dz}{4\pi\varepsilon_o\left(\rho^2 + z^2\right)^{3/2}}\hat{\rho}$$

$$= \frac{\rho_l\, \rho}{4\pi\varepsilon_o}\left[\frac{z}{\rho^2\sqrt{\rho^2 + z^2}}\right]_{-\infty}^{\infty}\hat{\rho}$$

$$= \frac{\rho_l}{2\pi\varepsilon_o \rho}\hat{\rho}$$

Distribution of Charges (5)

Electric Field of a Sheet Charge



$$d\vec{E} = \frac{dQ}{4\pi\varepsilon_o R^2}\left(\frac{-\rho\hat{\rho} + z\hat{z}}{\sqrt{\rho^2 + z^2}}\right)$$

$$= \frac{\rho_s \rho d\rho d\phi}{4\pi\varepsilon_o\left(\sqrt{\rho^2 + z^2}\right)^2}\left(\frac{-\rho\hat{\rho} + z\hat{z}}{\sqrt{\rho^2 + z^2}}\right)$$

The component radial, *ρ* is cancel out, because of all direction of component radial *ρ* around z

$$\vec{E} = \int_0^{2\pi}\int_0^{\infty} \frac{\rho_s \rho z d\rho d\phi}{4\pi\varepsilon_o\left(\rho^2 + z^2\right)^{3/2}}\hat{z}$$

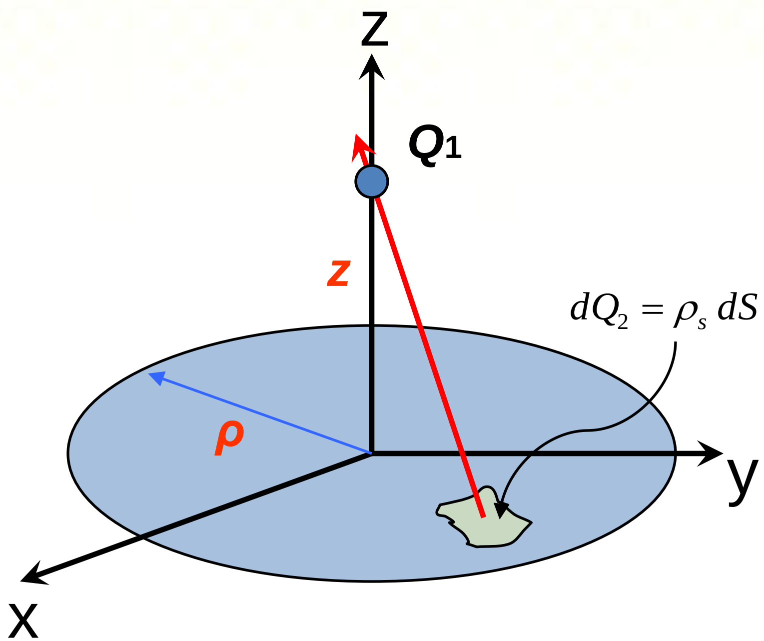
$$= \frac{\rho_s z}{2\varepsilon_o}\left[\frac{-1}{\sqrt{\rho^2 + z^2}}\right]_0^{\infty}\hat{z}$$

$$= \frac{\rho_s}{2\varepsilon_o}\hat{z}$$

Distribution of Charges (Example)

Electric Field of a Sheet Charge

Determine the force, *F* between the point charge, Q_1= 50 μC at (0, 0, 5) m and the disk charge, Q_2= 500 $\pi\mu$C with radial of ρ = 5 m and z = 0 m.



Step 1

$$\rho_s = \frac{Q_2}{A} = \frac{500\pi \times 10^{-6}\ \text{C}}{\pi\rho^2}$$

$$= \frac{500\pi \times 10^{-6}\ \text{C}}{\pi(5\text{m})^2}$$

$$= 0.2 \times 10^{-4}\ \text{C/m}^2$$

Step 2

$$\vec{R} = -\rho\hat{\rho} + 5\hat{z}$$

$$\hat{a}_R = \frac{-\rho\hat{\rho} + 5\hat{z}}{\sqrt{\rho^2 + 25}}$$

Step 3

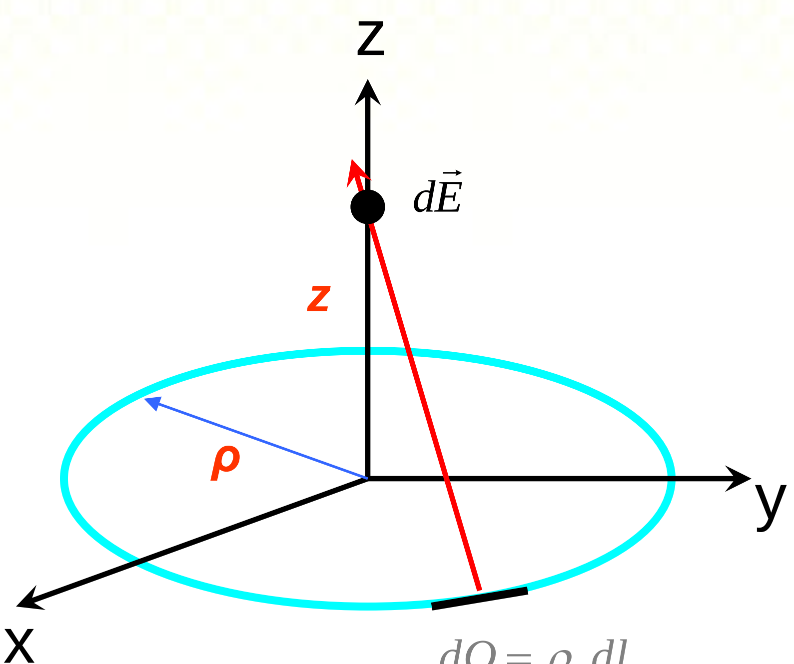
$$d\vec{F} = \frac{Q_1 dQ_2}{4\pi\varepsilon_o(\rho^2 + 25)}\left(\frac{-\rho\hat{\rho} + 5\hat{z}}{\sqrt{\rho^2 + 25}}\right)$$

$$= \frac{Q_1\rho_s \rho d\rho d\phi}{4\pi\varepsilon_o(\rho^2 + 25)^{3/2}}(-\rho\hat{\rho} + 5\hat{z})$$

$$\vec{F} = \int_0^{2\pi}\int_0^5 \frac{(50\times 10^{-6})(0.2\times 10^{-4})5\rho d\rho d\phi}{4\pi\varepsilon_o(\rho^2 + 25)^{3/2}}\hat{z}$$

$$= 16.56\hat{z}\ \text{Newton}$$

Distribution of Charges (6)
Electric Field of a Ring Charge



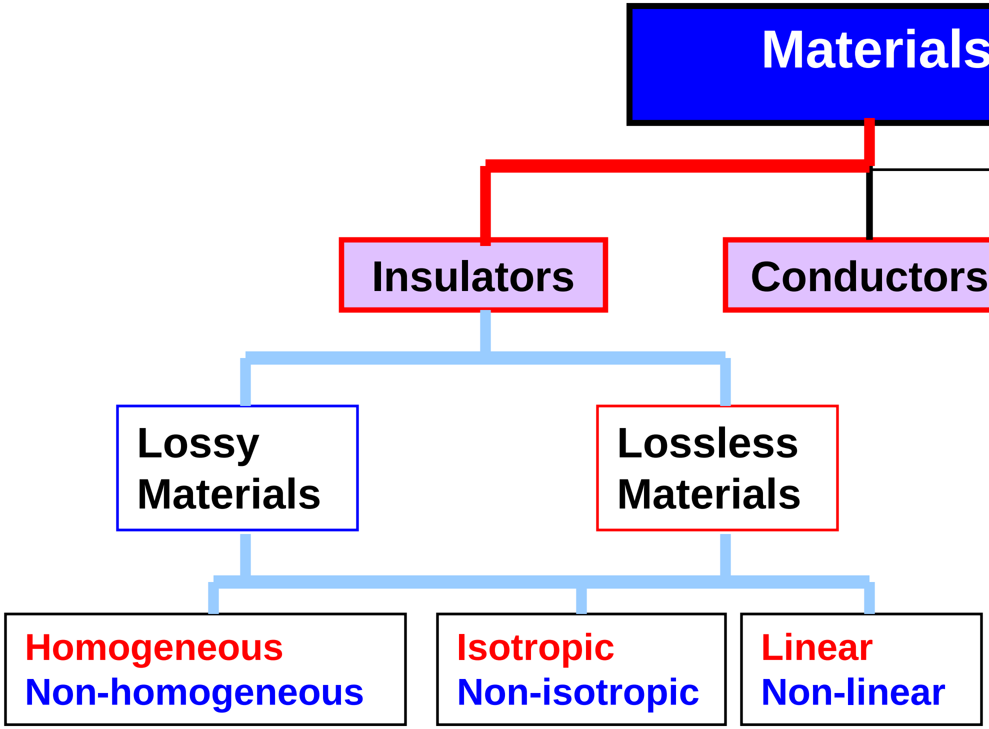
$$dQ = \rho_l \, dl = \rho_l \, \rho \, d\phi$$

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon_o R^2}\left(\frac{-\rho\hat{\rho} + z\hat{z}}{\sqrt{\rho^2 + z^2}}\right) = \frac{\rho_s \rho \, d\phi}{4\pi\varepsilon_o\left(\sqrt{\rho^2 + z^2}\right)^2}\left(\frac{-\rho\hat{\rho} + z\hat{z}}{\sqrt{\rho^2 + z^2}}\right)$$

The component radial, *ρ* is cancel out, because of all direction of component radial *ρ* around z

$$\vec{E} = \int_0^{2\pi} \frac{\rho_l \, \rho \, z \, d\phi}{4\pi\varepsilon_o\left(\rho^2 + z^2\right)^{3/2}} \hat{z} = \frac{\rho_l \, \rho \, z}{2\varepsilon_o\left(\rho^2 + z^2\right)^{3/2}} \hat{z}$$

Electrical Fields in Materials (1)



1. The electromagnetic **constitutive parameters** of a material medium are

a) **permittivity**, ε . (Electrical study)

b) **permeability**, μ. (Magnetic study)

c) **conductivity**, σ . (Electrical study)

Electrical Fields in Materials (Maxwell's Equations)

1. Modern electromagnetism is based on four fundamental relations

$$\vec{\nabla} \cdot \vec{D} = \rho_v$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

For an isotropic, linear and non-dispersive medium, the relations are

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

$$\vec{J} = \sigma \vec{E}$$ ← Ohm's law

2. In electrical study, we are concerned with only ε and σ

Electrical Fields in Materials

3. **A dielectric medium is linear if the magnitude of the induced polarization field is directly proportional to the magnitude of electric fields, $\vec{E}$.**

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$

where χ_e is called the electric susceptibility of the material.

4. **A dielectric medium is isotropic if the polarization field, $\vec{P}$ and electric field, $\vec{E}$ are in the same direction.**

5. **A dielectric medium is homogeneous if the ε, μ, and σ are constant throughout the medium.**

Conducting Materials (Conductors)

1. **Conductor is a material that easily conducts electrical current.**

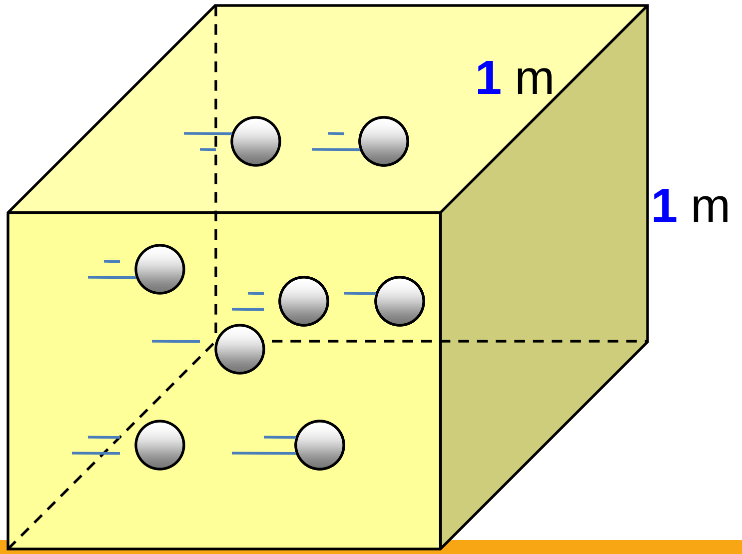
2. **Current through a given area is the electric charge passing through the area per unit time.**

3. **Current density, $\vec{J}$ is the current through a unit normal area.**

Example

8e charges across a unit area in 1 second

$$\vec{J} = 8e \ \mathrm{Am}^{-1}$$



Conducting Materials (Conductors)

4. In perfect dielectric, the conductivity, $\sigma = 0$

$$\vec{J} = 0$$

5. But, in perfect conductor, the conductivity, $\sigma = \infty$

$$\vec{E} = 0$$

6. Thus, perfect conductor cannot contain an electrostatic field within it.

7. The conductor is called an equipotential body, because the electric potential is the same at every point in the conductor.

Conducting Materials (Conductors)

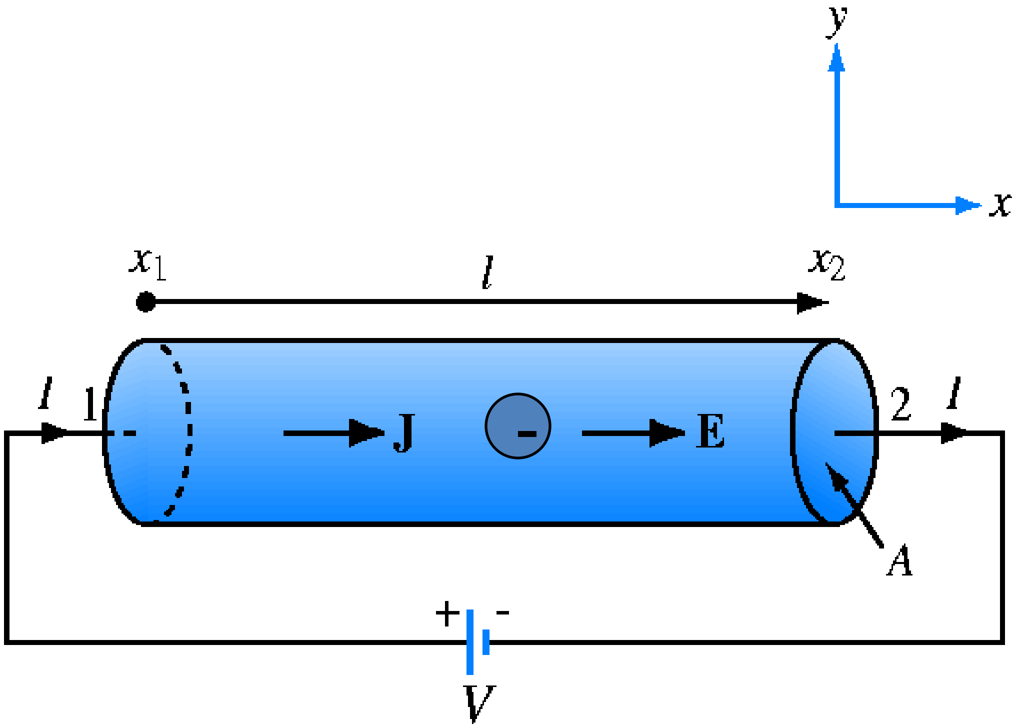
8. In general, conductor has resistivity, ρ_c because $\sigma \neq \infty$

Conductor	Conductivity, σ (S/m)
Silver	6.2×10^7
Copper	5.8×10^7
Gold	4.1×10^7
Aluminium	3.5×10^7

9. The relationship between conductivity, σ and resistivity, ρ_c

$$\rho_c = \frac{1}{\sigma}$$

Conducting Materials (Conductors)



For non-perfect conductor, the $\vec{E} \neq 0$ **, the resistance,** ***R*** **is occurred in the conductor**

$$R = \frac{V}{I}$$

$$= \frac{\int_v \vec{E} \cdot d\vec{l}}{\int_S \sigma \vec{E} \cdot d\vec{S}}$$

Dielectric Materials (Insulators)

1. There are two type of dielectric materials.

a) Lossless materials

b) Lossy materials

2. In general, the relative permittivity, ε_r of lossy materials consist of real and imaginary parts.

$$\varepsilon_r = \varepsilon_r' - j\frac{\sigma}{\omega}$$

3. The real part, ε_r' is related to the ability of the material to store electrical energy and the imaginary part, σ/ω is the energy-dissipating component.

4. For lossless materials, the $\sigma/\omega \approx 0$

5. The lossy medium can be polarized by an external electric field, $\vec{E}$

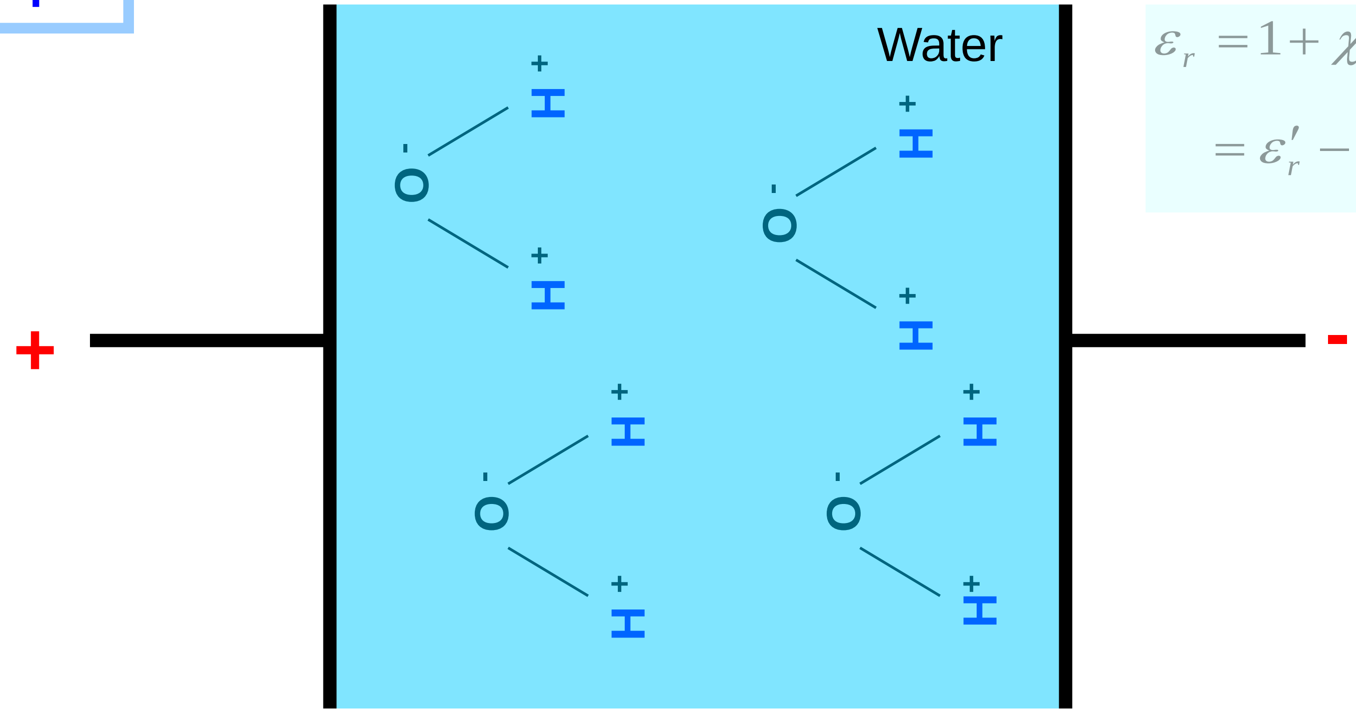
Dielectric Materials (Insulators)

8. The electric flux density, $\vec{D}$ in a lossy medium is written as

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P}$$ ← **Polarization vector**

where $\vec{P} = \varepsilon_o \chi_e \vec{E}$ and χ_e is called the electric susceptibility of the material.

Example



$$\varepsilon_r = 1 + \chi_e$$
$$= \varepsilon_r' - j\frac{\sigma}{\omega}$$

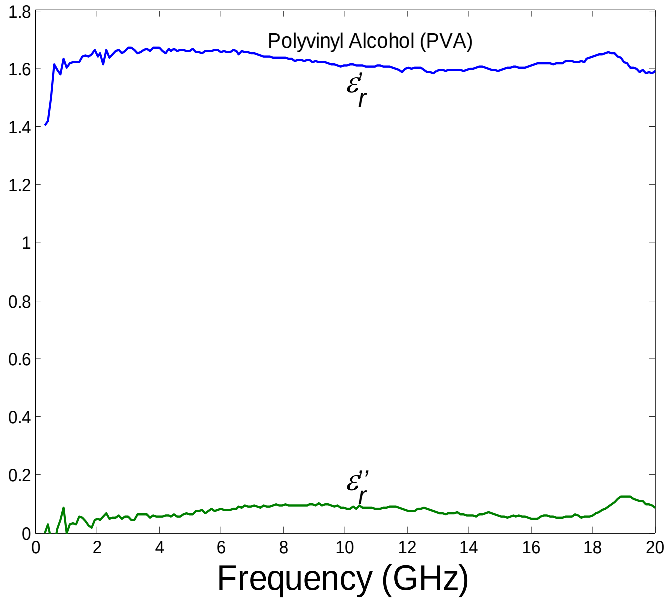
Dielectric Materials (Insulators)

9. The electric susceptibility, χ_e is the maximum electric field that a dielectric can tolerate or withstand without electrical breakdown.

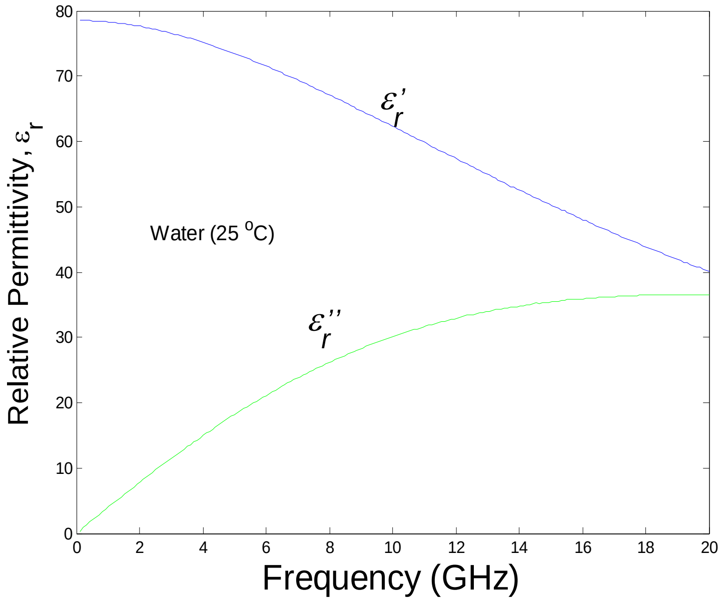
10. Dielectric breakdown occurred when a dielectric becomes conducting.

Example

Lossless Materials (PVA)

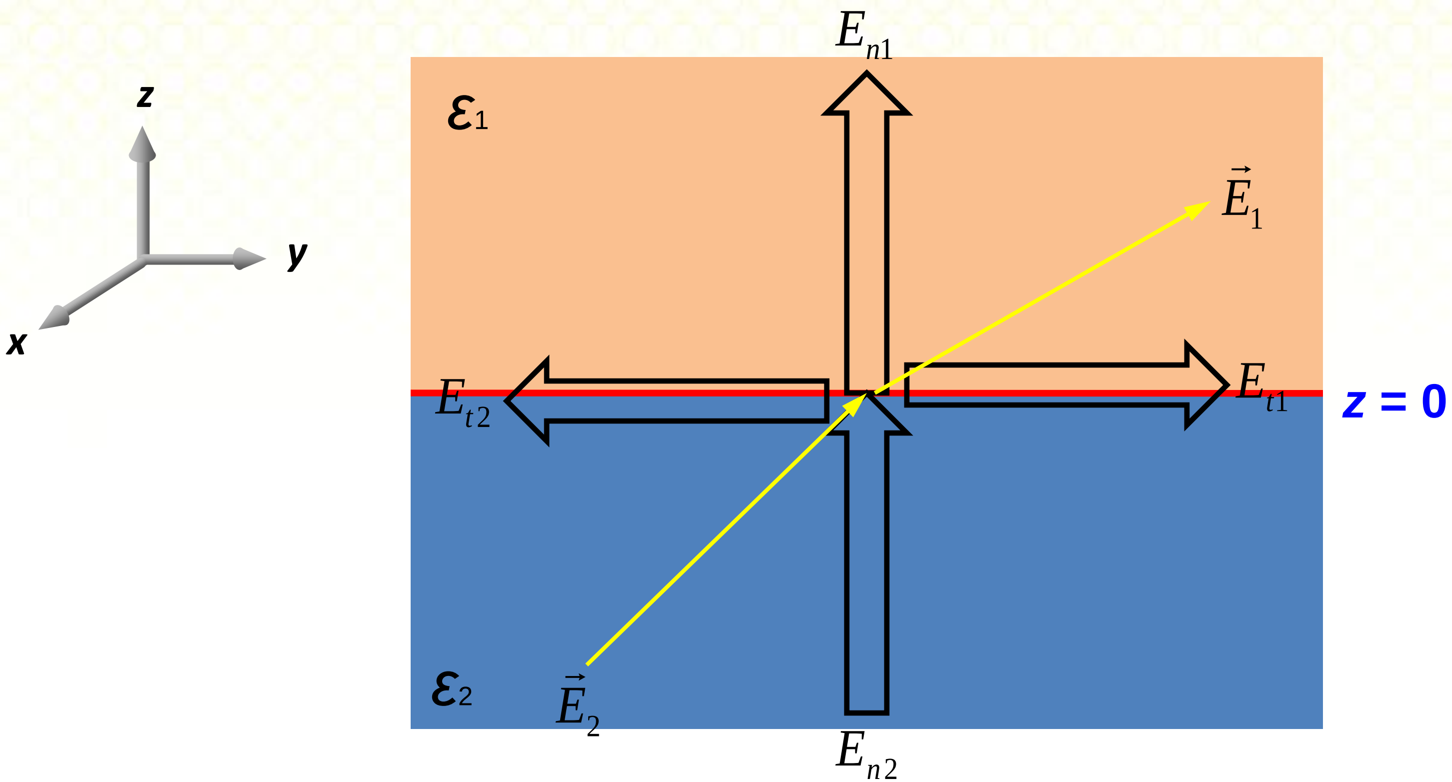


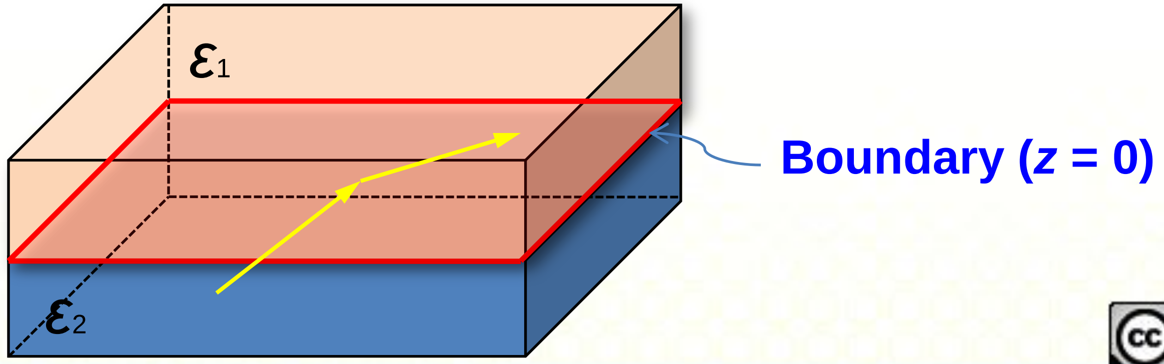
Lossy Materials (Water)



BOUNDARY CONDITIONS

Two Extensive Homogeneous Isotropic Dielectric





BOUNDARY CONDITIONS (1)

1) Tangential $\vec{E}$ is always continuous.

$$E_{t1} = E_{t2}$$

2) Tangential $\vec{H}$ is continuous.

$$H_{t1} = H_{t2}$$

Tangential $\vec{H}$ is discontinuous by an amount corresponding to any surface current, $\vec{J}_s$ which may flow.

$$H_{t1} = H_{t2} + \vec{J}_s$$

3) Normal $\vec{B}$ is always continuous.

$$B_{n1} = B_{n2}$$

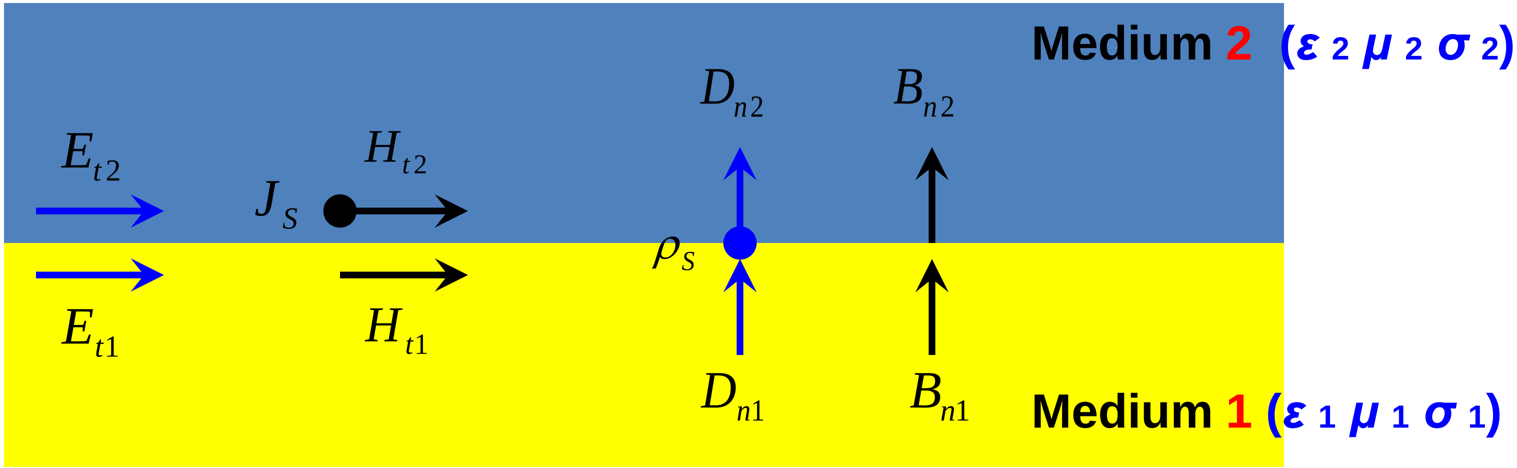
BOUNDARY CONDITIONS (2)

4) Normal $\vec{D}$ is continuous.

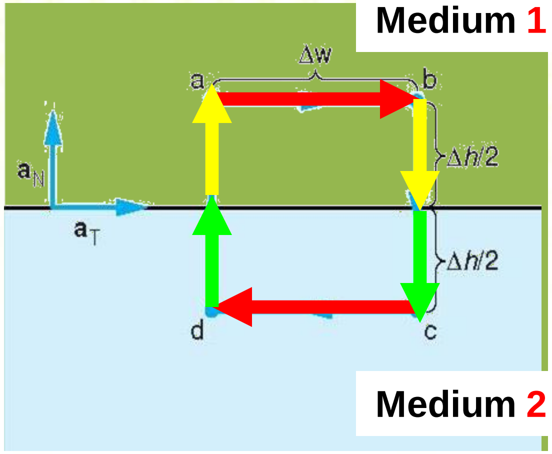
$$D_{n1} = D_{n2}$$

Normal $\vec{D}$ is discontinuous by an amount corresponding to any surface charge, ρ_s which may be present.

$$D_{n1} = D_{n2} + \rho_s$$



BOUNDARY CONDITIONS
Proof (3)



For static fields,

$$\oint \vec{E}\cdot d\vec{l} = 0$$

Integrate in the loop clockwise starting from *a*,

$$\int_a^b \vec{E}\cdot d\vec{l} + \int_b^c \vec{E}\cdot d\vec{l} + \int_c^d \vec{E}\cdot d\vec{l} + \int_d^a \vec{E}\cdot d\vec{l} = 0$$

Evaluate each segment,

$$\int_a^b \vec{E}\cdot d\vec{l} = \int_0^{\Delta w} E_{T1}\mathrm{a}_T \cdot dl\mathrm{a}_T = E_{T1}\Delta w$$

$$\int_c^d \vec{E}\cdot d\vec{l} = \int_{\Delta w}^{0} E_{T2}\mathrm{a}_T \cdot dl\mathrm{a}_T = -E_{T2}\Delta w$$

$$\int_b^c \vec{E}\cdot d\vec{l} = \int_{\Delta h/2}^{0} E_{N1}\mathrm{a}_N \cdot dl\mathrm{a}_N + \int_0^{-\Delta h/2} E_{N2}\mathrm{a}_N \cdot dl\mathrm{a}_N$$

$$= -(E_{N1}+E_{N2})\frac{\Delta h}{2}$$

$$\int_d^a \vec{E}\cdot d\vec{l} = \int_{-\Delta h/2}^{0} E_{N2}\mathrm{a}_N \cdot dl\mathrm{a}_N + \int_0^{\Delta h/2} E_{N1}\mathrm{a}_N \cdot dl\mathrm{a}_N$$

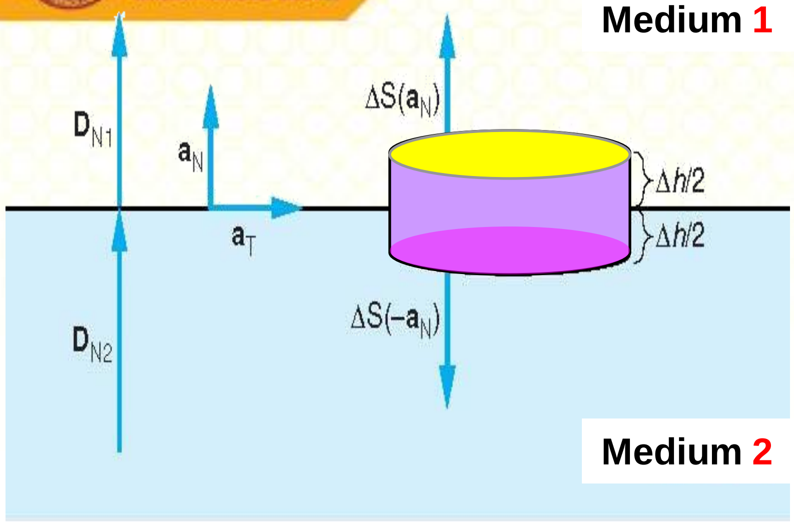
$$= (E_{N1}+E_{N2})\frac{\Delta h}{2}$$

Summing for each segment, then we have the first boundary condition:

$$E_{T1}\Delta w - E_{T2}\Delta w = 0$$

$$E_{T1} = E_{T2}$$

BOUNDARY CONDITIONS
Proof (4)



The Gauss's Law,

$$\oint \vec{D} \cdot dS = Q_{enc}$$

Thus,

$$\oint \vec{D} \cdot d\vec{S} = \int_{top} \vec{D} \cdot d\vec{S} + \int_{bottom} \vec{D} \cdot d\vec{S} + \int_{side} \vec{D} \cdot d\vec{S}$$

The pillbox is short enough, so the flux passes through the side is negligible.

$$\int_{top} \vec{D} \cdot d\vec{S} = \int D_{N1} a_N \cdot dS a_N = D_{N1} \Delta S$$

$$\int_{bottom} \vec{D} \cdot d\vec{S} = \int D_{N2} \mathrm{a}_N \cdot dS(-\mathrm{a}_N) = -D_{N2} \Delta S$$

Which sums to

$$(D_{N1} - D_{N2})\Delta S = Q_{enc}$$

Thus, it leads to the second boundary condition

$$D_{N1} - D_{N2} = \rho_S$$

Questions

Two extensive homogeneous isotropic dielectric meet on plane z=0. For *z* > 0, $\varepsilon_{r1} = 4$ and z < 0, $\varepsilon_{r2} = 3$. An uniform electric field, $\vec{E}_1 = 5\hat{a}_x - 2\hat{a}_y + 3\hat{a}_z$ kV/m exists for $z \geq 0$

Find

a) E_2 for $z \leq 0$

b) The angles E_1 and E_2 make with the interface

c) The energy densities in both dielectrics

d) The energy within a cube of side 2 m centered at (3, 4, -5)

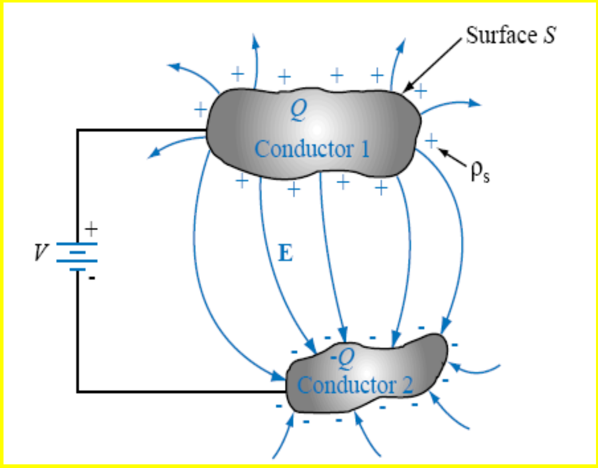
Capacitance (1)

1. The amount of charge, ***Q*** that accumulates as a function of potential difference, ***V*** is called the **capacitance, *C***.

$$C = \frac{Q}{V}$$

2. The unit **capacitance** is the **farad** (F) or coulomb per volt.

3. **Capacitor** can be created using two conducting bodies separated by an dielectric (insulator) medium.



Capacitance (2)

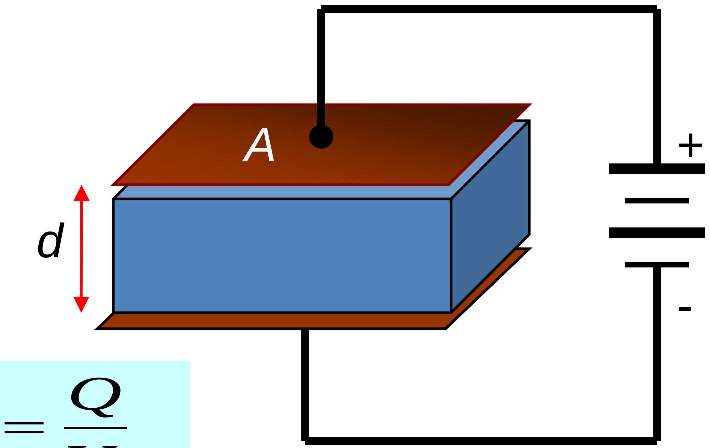
4. The three general form of capacitors are

a) Parallel-plate capacitor

b) Coaxial capacitor

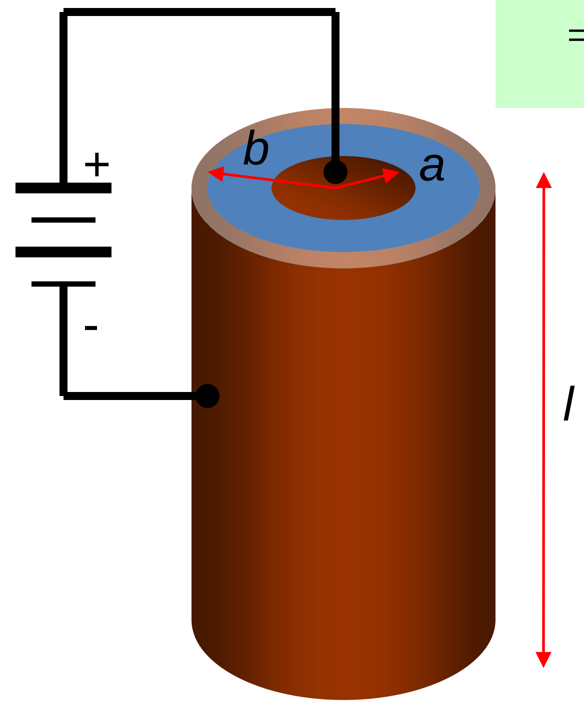
c) Spherical capacitor

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon l}{\ln(b/a)}$$



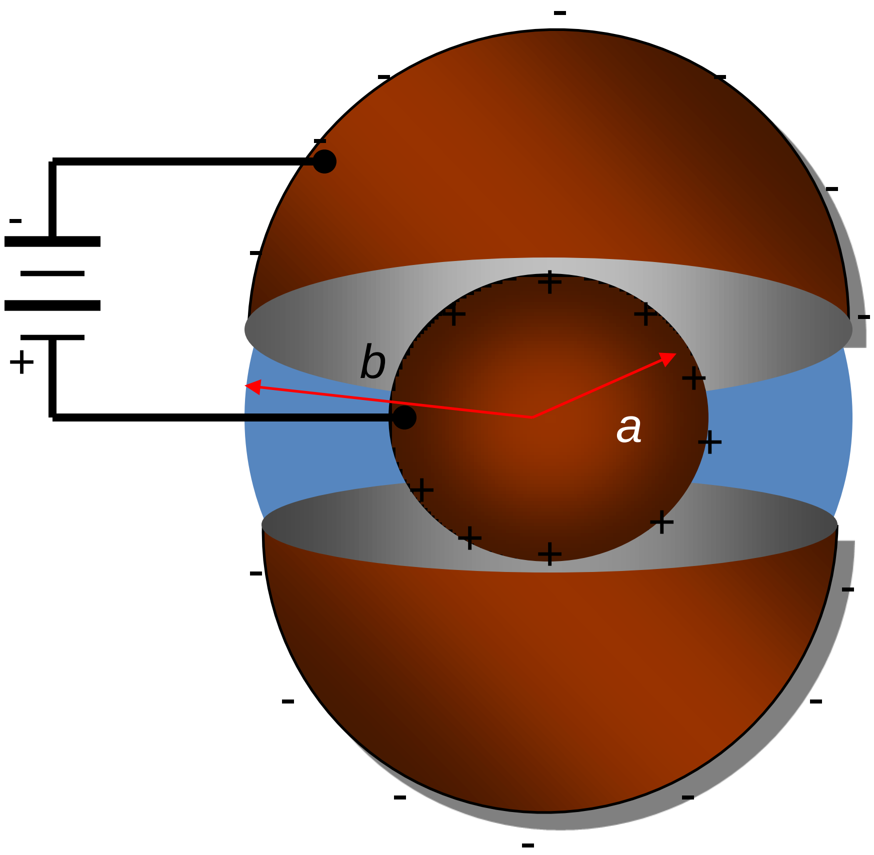
$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\varepsilon A}{d}$$

Parallel-plate capacitor



Coaxial capacitor

Capacitance (3)



Spherical capacitor

$$C = \frac{Q}{V}$$

$$= \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

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