SKF 3143

Process Control and Dynamics:

Response of Second Order Systems

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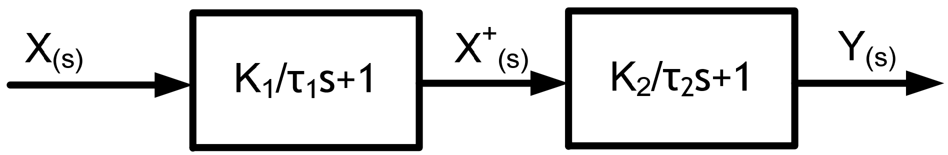
Learning Objectives

When I complete this chapter, I want to be able to do the following:

1. Analyze the response of the second order systems

Second-Order Systems

- A second-order transfer function can arise physically whenever two first-order processes are connected in series.
- For example, two stirred-tank heaters, each with first-order transfer function relating inlet to outlet temperature, might be physically connected so the outflow stream of the first heater is used as the inflow stream of the second tank.



- Here

$$G_{(s)} = \frac{Y_{(s)}}{X_{(s)}} = \frac{K_1 K_2}{(\tau_1 s + 1)(\tau_2 s + 1)} = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)} \qquad (7.1)$$

- In this chapter we consider the case where the second-order transfer function has the standard form

$$G_{(s)} = \frac{K}{\tau^2 s^2 + 2\zeta \tau s + 1} \qquad (7.2)$$

Where ζ (zeta)= damping coefficient (dimensionless)

Second-Order Systems

- The type of behavior that occur depends on the numerical value of damping coefficient, ζ:

  It is convenient to consider three types of behavior:

Damping Coefficient	*Types of Response*	*Root of Characteristic Polynomial*
$\zeta > 1$	Overdamped	Real and unequal
$\zeta = 1$	Critically damped	Real and equal
$0 < \zeta < 1$	Underdamped	Complex conjugates

- What about $\zeta < 0$? It results in unstable system
- The characteristic polynomial is the denominator of the transfer function:

$$\tau^2 s^2 + 2\zeta\tau s + 1$$

Second-Order Systems

- The transfer function given by Eqs. 7.1 and 7.2 differ only in the denominators.
- Equating them yields the relation between the two alternative forms for the *overdamped* second-order case.
- Note that when $\zeta \geq 1$, the denominator of 7.2 can be factored as:

$$\tau^2 s^2 + 2\zeta\tau s + 1 = (\tau_1 s + 1)(\tau_2 s + 1) \qquad (7.3)$$

- Expanding the right side of 7.3 and equating coefficients of the *s* terms, yields

$$\tau^2 = \tau_1 \tau_2$$

$$2\zeta\tau = \tau_1 + \tau_2$$

from which we obtain

$$\tau = \sqrt{\tau_1 \tau_2}$$

$$\zeta = \frac{\tau_1 + \tau_2}{2\sqrt{\tau_1 \tau_2}}$$

Second-Order Systems

- Alternatively, the left side of 7.3 can be factored:

$$\tau^2 s^2 + 2\zeta\tau s + 1 = \left(\frac{\tau s}{\zeta - \sqrt{\zeta^2 - 1}} + 1\right)\left(\frac{\tau s}{\zeta + \sqrt{\zeta^2 - 1}} + 1\right) \qquad (7.4)$$

from which expression for τ_1 and τ_2 are obtained

$$\tau_1 = \frac{\tau}{\zeta - \sqrt{\zeta^2 - 1}} \qquad (\zeta \geq 1) \qquad (7.5)$$

$$\tau_2 = \frac{\tau}{\zeta + \sqrt{\zeta^2 - 1}} \qquad (\zeta \geq 1) \qquad (7.6)$$

Response of Second-Order Systems

Step Response

- For the step input with transform:

$$Y_{(s)} = \frac{KM}{s\left(\tau^2 s^2 + 2\zeta\tau s + 1\right)} \qquad (7.7)$$

- After some manipulation, and inverting to the time domain, three forms of response are obtained:

Case a (ζ > 1)

If the denominator of Eq. 7.7 is factored using Eqs. 7.5 and 7.6, then the response can be written

$$y_{(t)} = KM\left(1 - \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{\tau_1 - \tau_2}\right) \qquad (7.8)$$

If the denominator of Eq. 7.7 is left unfactored, then the response can be written in the equivalent form

$$y_{(t)} = KM\left\{1 - e^{-\zeta t/\tau}\left[\cosh\left(\frac{\sqrt{\zeta^2 - 1}}{\tau}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sinh\left(\frac{\sqrt{\zeta^2 - 1}}{\tau}t\right)\right]\right\} \qquad (7.9)$$

Response of Second-Order Systems

Case b (ζ = 1)

$$y_{(t)} = KM\left[1-\left(1+\frac{t}{\tau}\right)e^{-t/\tau}\right] \qquad (7.10)$$

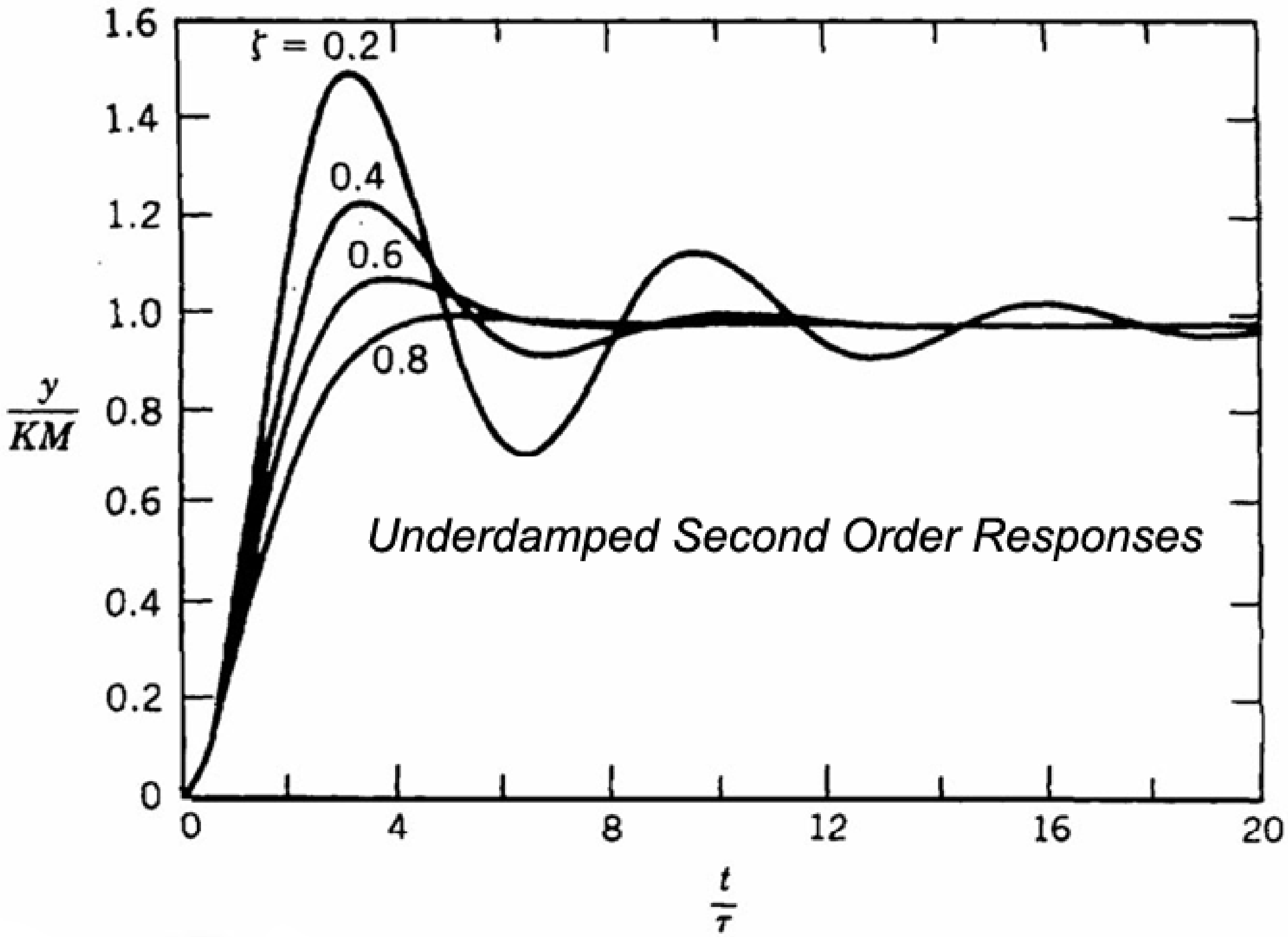
Case c (0 ≤ ζ < 1)

$$y_{(t)} = KM\left\{1-e^{-\zeta t/\tau}\left[\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)\right]\right\} \qquad (7.11)$$

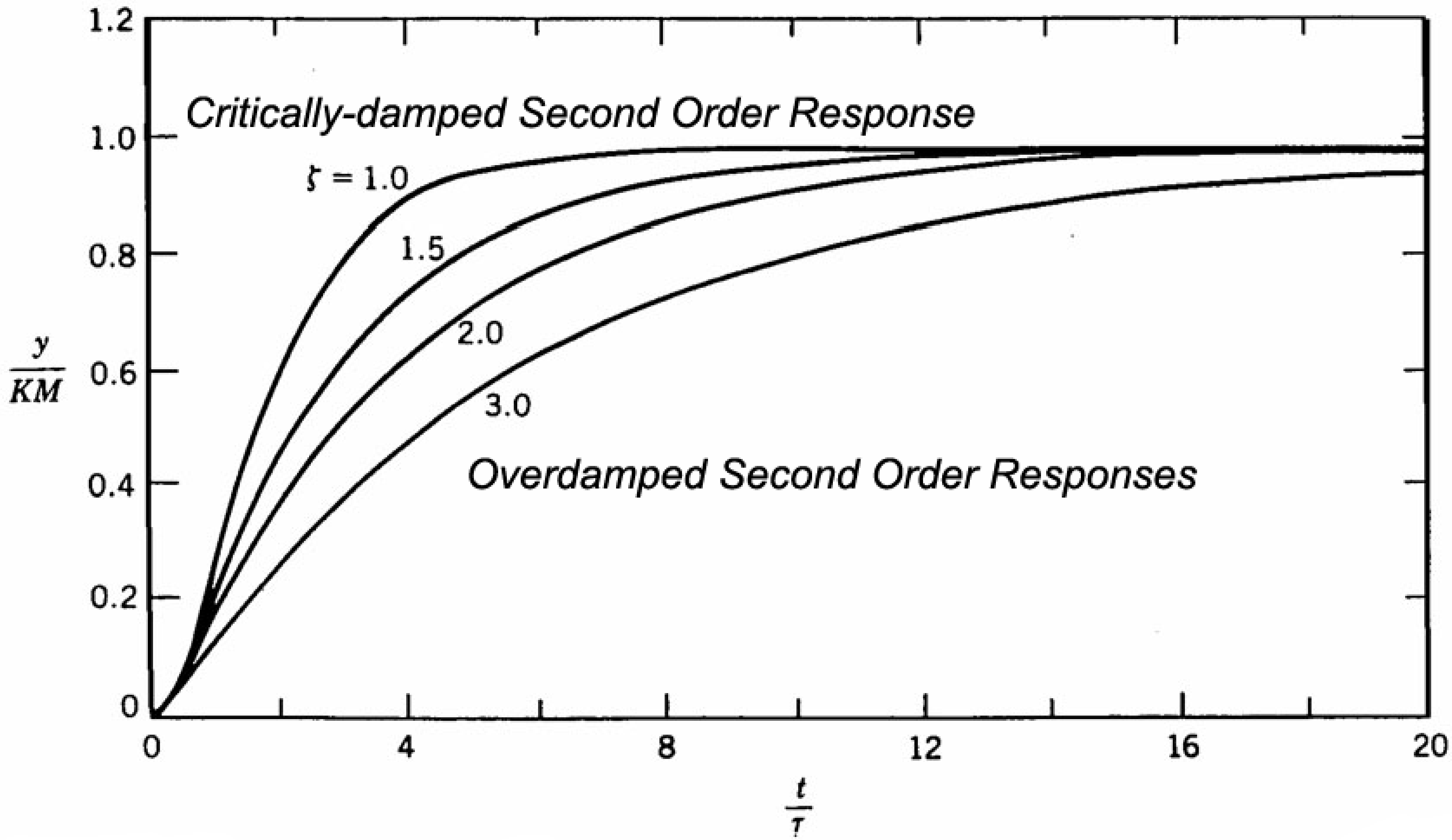
Several general remarks can be made concerning the responses shown in Figs. 7.1 and 7.2:

1. Responses exhibiting oscillation and overshoot ($y/KM > 1$) are obtained only for values of ζ less than one.
2. Large values of ζ yield a sluggish (slow) response.
3. The fastest response without overshoot is obtained for the critically damped case (ζ = 1).

Step response of underdamped second-order systems



Step response of critically-damped and overdamped second-order systems



Response of Second-Order Systems

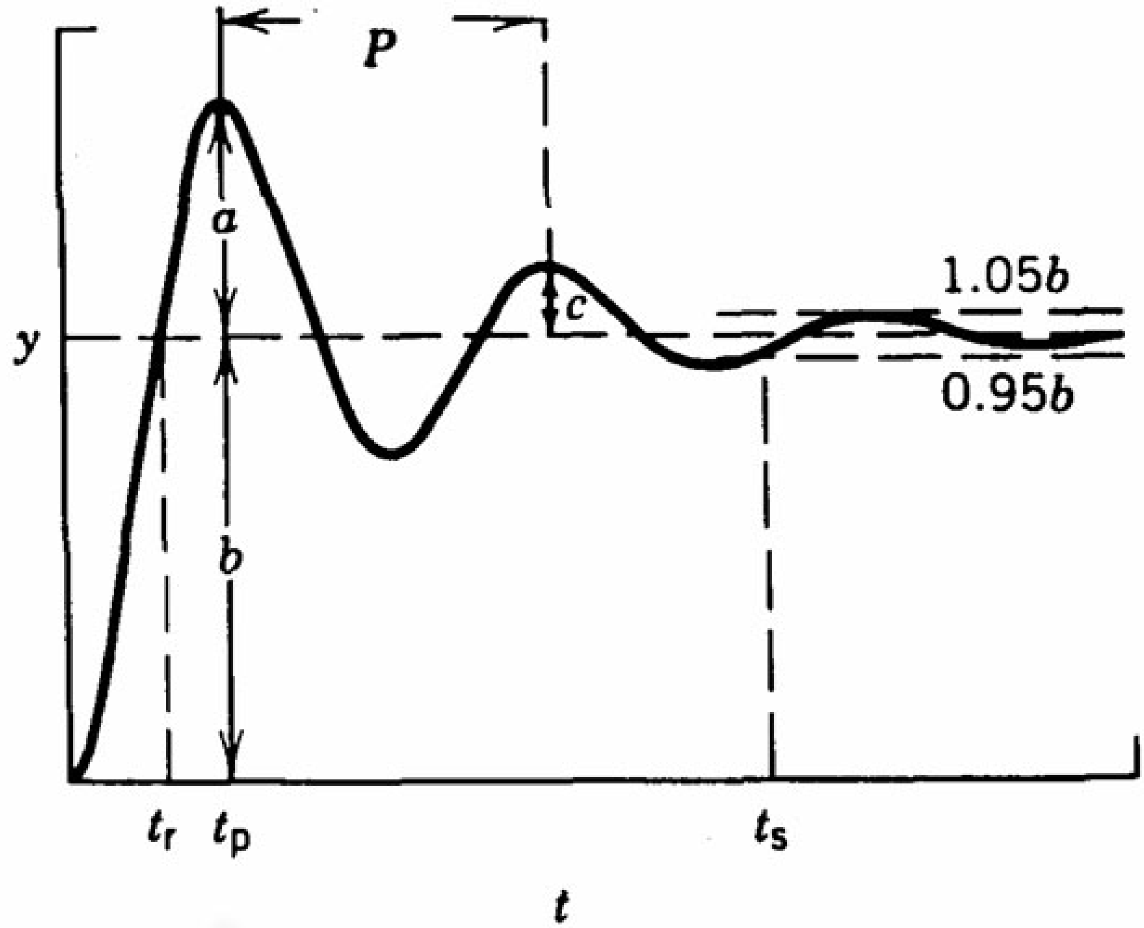
- Control system designers often attempt to make the setpoint step response of the controlled variable approximate the step response of underdamped second-order system.

- That is, make it exhibit a prescribed amount of overshoot and oscillation as it settles at the new operating point.

- Values of ζ in the range 0.4 to 0.8 often are suitable for specifying a desired control system response, assuming that it can be approximated as an underdamped second-order system.

- In this range, the controlled variable *y* reaches the new operating point faster than with $\zeta = 1$ or 1.5, but the response is much less oscillatory (it settles faster) than with $\zeta = 0.2$.

Performance characteristics for the step response of an underdamped process.



Underdamped Second Order Response

Response of Second-Order Systems

1. **Rise Time.** t_t is the time the process output takes to first reach the new steady-state value.
2. 2. **Time to First Peak.** t_p is the time required for the output to reach its first maximum value.
3. **Settling Time.** t_s is defined as the time required for the process output to reach and remain inside a band whose width is equal ±5% of the total change in *y*. The term 95% response time sometimes is used to refer to this case. Also, values of ±1% sometimes are used.
4. **Overshoot.** OS = *a/b* (% overshoot is 100*a/b*).
5. **Decay Ratio.** DR = *c/a* (where *c* is the height of the second peak).
6. **Period of Oscillation.** *P* is the time between two successive peaks or two successive valleys of the response.

References:

- Seborg, D. E., Edgar, T. F., Mellinchamp, D. A. (2003). *Process Dynamics and Control*, 2nd. Edition. John Wiley, ISBN: 978-04-71000-77-8.
- Marlin, T. E. (2000). *Process Control: Designing Processes and Control System for Dynamic Performance*, 2nd. Edition. McGraw Hill, ISBN: 978-00-70393-62-2.
- Stephanopoulos, G. (1984). *Chemical Process Control. An Introduction to Theory and Practice*. Prentice Hall, ISBN: 978-01-31286-29-0