SKF 3143

Process Control and Dynamics:

Transfer Function

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Learning Objectives

When I complete this chapter, I want to be able to do the following:

1. Develop transfer function of process control

Transfer Function

- Transfer function = G(s)

$$= \frac{\text{Laplace Transform of output variables in deviation form}}{\text{Laplace Transform of input variables in deviation form}}$$

- It describes completely the dynamic behaviour of the output when the corresponding input changes are given.

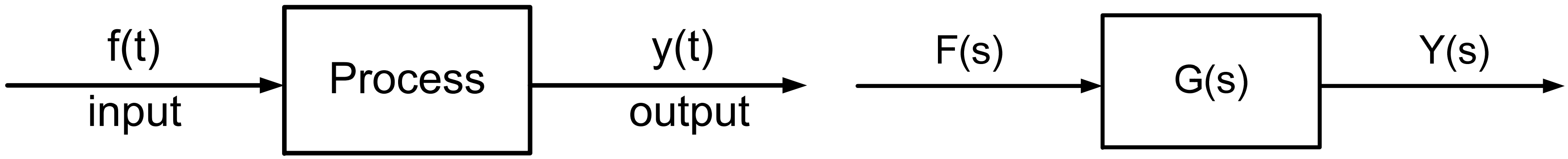
$$G_{(s)} = \frac{K}{\tau s + 1}$$

K = steady-state gain

τ = system time constant

Input-Output Model

- For a system with a single input and a single output, the dynamic behaviour of the process us described by an *n* th order linear differential equation



$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = bf(t)$$

$$y(0) = \left[\frac{dy}{dt}\right]_{t=0} = \left[\frac{d^2 y}{dt^2}\right]_{t=0} = \cdots = \left[\frac{d^{n-1} y}{dt^{n-1}}\right]_{t=0} = 0$$

$$\frac{Y_{(s)}}{F_{(s)}} = G_{(s)} = \frac{b}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

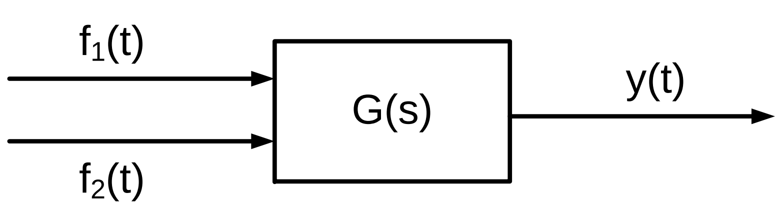
Process With Two Inputs

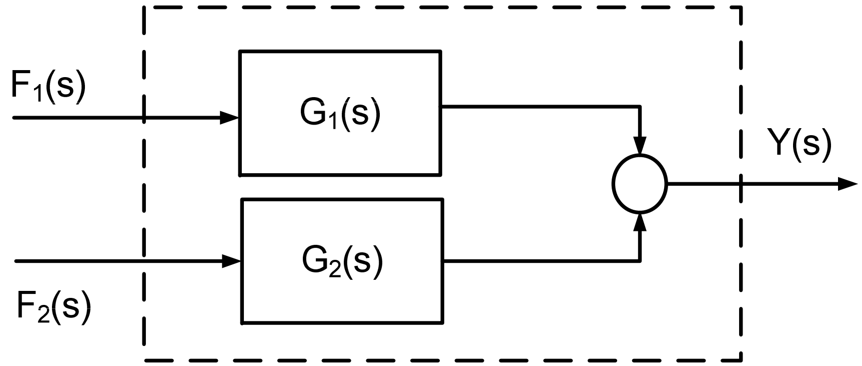
- For a dynamic model of two inputs $f_1(t)$ and $f_2(t)$

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_1 f_1(t) + b_2 f_2(t)$$

$$Y_{(s)} = \frac{b_1}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} F_{1(s)} + \frac{b_2}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} F_{2(s)}$$

$$Y_{(s)} = G_{1(s)} F_{1(s)} + G_{2(s)} F_{2(s)}$$

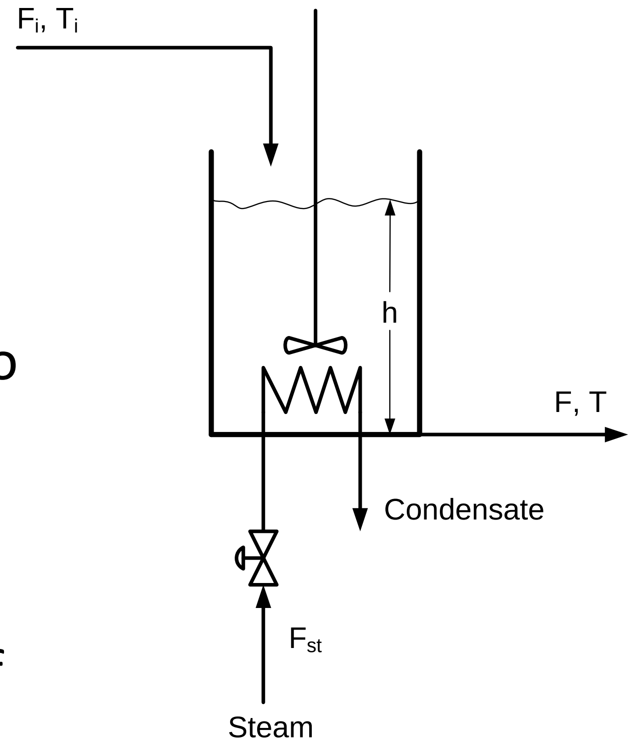




Development of Transfer Function

- Consider a simple first-order differential equation derived for the stirred-tank heating system:

$$V\frac{dT}{dt} = F(T_i - T) + \frac{Q}{\rho C_p}$$



- Assume volume of the tank is constant, therefore F_i = F. The value of ρ and C_p are also constant.
- At time zero the system is at its steady state; hence T(0)=0.
- Taking the Laplace transform of both sides of the equation, we have

$$VL\left(\frac{dT}{dt}\right) = L\left[F(T_i - T) + \frac{Q}{\rho C_p}\right]$$

Development of Transfer Function

$$VL\left(\frac{dT}{dt}\right) = FL(T_i) - FL(T) + \frac{1}{\rho C_p} L(Q)$$

- The constant has been factored out of the transform. Since T(t), T_i(t), and Q(t) are unspecified, their transforms can be expressed in a general manner:

$$VsT_{(s)} = FT_{i(s)} - FT_{(s)} + \frac{1}{\rho C_p} Q_{(s)}$$

- Rearranging gives

$$(Vs + F)T_{(s)} = FT_{i(s)} + \frac{1}{\rho C_p} Q_{(s)}$$

$$T_{(s)} = \left(\frac{F}{Vs + F}\right) T_{i(s)} + \left(\frac{1/\rho C_p}{Vs + F}\right) Q_{(s)}$$

$$T_{(s)} = \left(\frac{1}{V/F\, s + 1}\right) T_{i(s)} + \left(\frac{1/F\rho C_p}{V/F\, s + 1}\right) Q_{(s)}$$

Development of Transfer Function

$$T_{(s)} = G_{1(s)} T_{i(s)} + G_{2(s)} Q_{(s)}$$

- $G_{1(s)}$ and $G_{2(s)}$ are called *transfer functions*.
- $G_{1(s)}$ relates the input $T_{i(s)}$ to the output $T_{(s)}$; $G_{2(s)}$ has similar role for input $Q_{(s)}$.

$$\frac{T_{(s)}}{T_{i(s)}} = G_{1(s)} = \frac{1}{V/F\, s + 1}$$

$$K_1 = 1$$

$$\tau_1 = \frac{V}{F}$$

$$\frac{T_{(s)}}{Q_{(s)}} = G_{2(s)} = \frac{1/F\rho C_p}{V/F\, s + 1}$$

$$K_2 = \frac{1}{F\rho C_p}$$

$$\tau_2 = \frac{V}{F}$$

Step Changed Input

- At steady-state

$$T'_{(s)} = \left(\frac{1}{V/_F\, s+1} \right) T'_{i(s)} + \left(\frac{1/_{F\rho C_p}}{V/_F\, s+1} \right) Q'_{(s)}$$

- Assume the inlet temperature is held constant (T_i = T_i), then $T'_{i(s)}$=0.
- Suppose the heat input is changed by a step input at t=0 from its value of Q to a new value, Q+ΔQ. Therefore, Q'=ΔQ for t≥0.
- Use Table 3.1 to obtain $Q'_{(s)}$=ΔQ/s

$$T'_{(s)} = \left(\frac{1/_{F\rho C_p}}{V/_F\, s+1} \right) \frac{\Delta Q}{s}$$

- Observe form Table 3.1 that $T'_{(s)}$ corresponds to the time domain function

$$T'_{(t)} = K\Delta Q\left(1 - e^{-t/_\tau}\right) = \frac{1}{F\rho C_p}\Delta Q\left(1 - e^{-Ft/_V}\right)$$

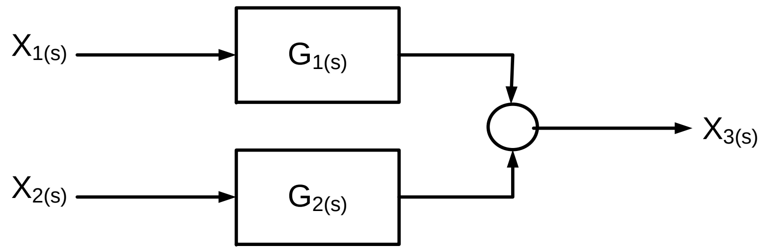
Steady-state response

Properties of Transfer Functions

Additive Property

- A general form is

$$X_3 = G_{1(s)} X_{1(s)} + G_{2(s)} X_{2(s)}$$



- In figure above observe that a single process output variable (X_3) may be influenced by more than one input (X_1 and X_2) acting singly or together.
- In such a case the total output change is calculated by summing the individual input contributions in the s-domain before inverting to the time domain.
- In the case, $X_{3(s)}$ is the composite output response that results from both input dynamic effects, $X_{1(s)}$ and $X_{2(s)}$.

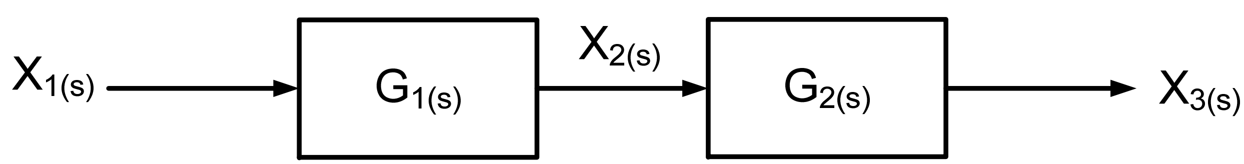
Properties of Transfer Functions

<u>Multiplicative Property</u>

- Transfer function also exhibit a multiplicative property for sequential processes or process elements.
- Suppose two processes with transfer functions G_1 and G_2 are placed in series.
- The input $X_{1(s)}$ to G_1 yields an output $X_{2(s)}$, which is the input to G_2. The output from G_2 is X_3.
- In equation form

$$X_{2(s)} = G_{1(s)} X_{1(s)}$$

$$X_{3(s)} = G_{2(s)} X_{2(s)} = G_{2(s)} G_{1(s)} X_{1(s)}$$



- In other words, the transfer function between the original input X_1 and the output X_3 can be found by multiplying G_2 by G_1.

Linearization

- We must convert the rigorous nonlinear differential equations describing a chemical system into linear differential equations so that we can use the powerful linear mathematical techniques.
- What is a linear differential equation?
- Basically, it is one that contains variables only to the first power in any one term of the equation.
- If square roots, squares, exponentials, products of variables, etc. appear in the equation, it is nonlinear.
- Linear example:

$$a_1 \frac{dx}{dt} + a_0 x = f_{(t)}$$

where a_0 and a_1 are constants or functions of time only, not of dependent variables or their derivatives.

Linearization

- Nonlinear examples:

$$a_1 \frac{dx}{dt} + a_0 x^{0.5} = f_{(t)}$$

$$a_1 \frac{dx}{dt} + a_0 (x)^2 = f_{(t)}$$

$$a_1 \frac{dx}{dt} + a_0 e^x = f_{(t)}$$

$$a_1 \frac{dx_1}{dt} + a_0 x_{1(t)} x_{2(t)} = f_{(t)}$$

where x_1 and x_2 are both dependent variables.

- Mathematically, a linear differential equations is one for which the following two properties hold:

1. If $x_{(t)}$ is a solution, then $cx_{(t)}$ is also a solution, where c is a constant.
2. If x_1 is a solution and x_2 is also a solution, then x_1+x_2 is a solution.

Linearization

- Linearization is quite straightforward.
- All we do is take the nonlinear functions, expand them in Taylor series around the steady-state operating level, and neglect all terms after the first partial derivatives.
- Lets assume we have a nonlinear function *f* of the process variables x_1 and x_2: $f(x_1,x_2)$.
- For example, x_1 could be mole fraction or temperature or flow rate.
- We will denote the steady-state values of these variables by using an overscore:

$\overline{x_1} \equiv$ steady-state value of x_1

$\overline{x_2} \equiv$ steady-state value of x_2

Linearization

- Now we expand the function $f_{(x1,x2)}$ around its steady-state value

$$f_{(x_1,x_2)} = f_{(\overline{x_1},\overline{x_2})} + \left(\frac{\partial f}{\partial x_1}\right)_{(\overline{x_1},\overline{x_2})}\left(x_1 - \overline{x_1}\right) + \left(\frac{\partial f}{\partial x_2}\right)_{(\overline{x_1},\overline{x_2})}\left(x_2 - \overline{x_2}\right)$$
$$+ \left(\frac{\partial^2 f}{\partial x_1^2}\right)_{(\overline{x_1},\overline{x_2})}\frac{\left(x_1 - \overline{x_1}\right)^2}{2!} + \cdots$$

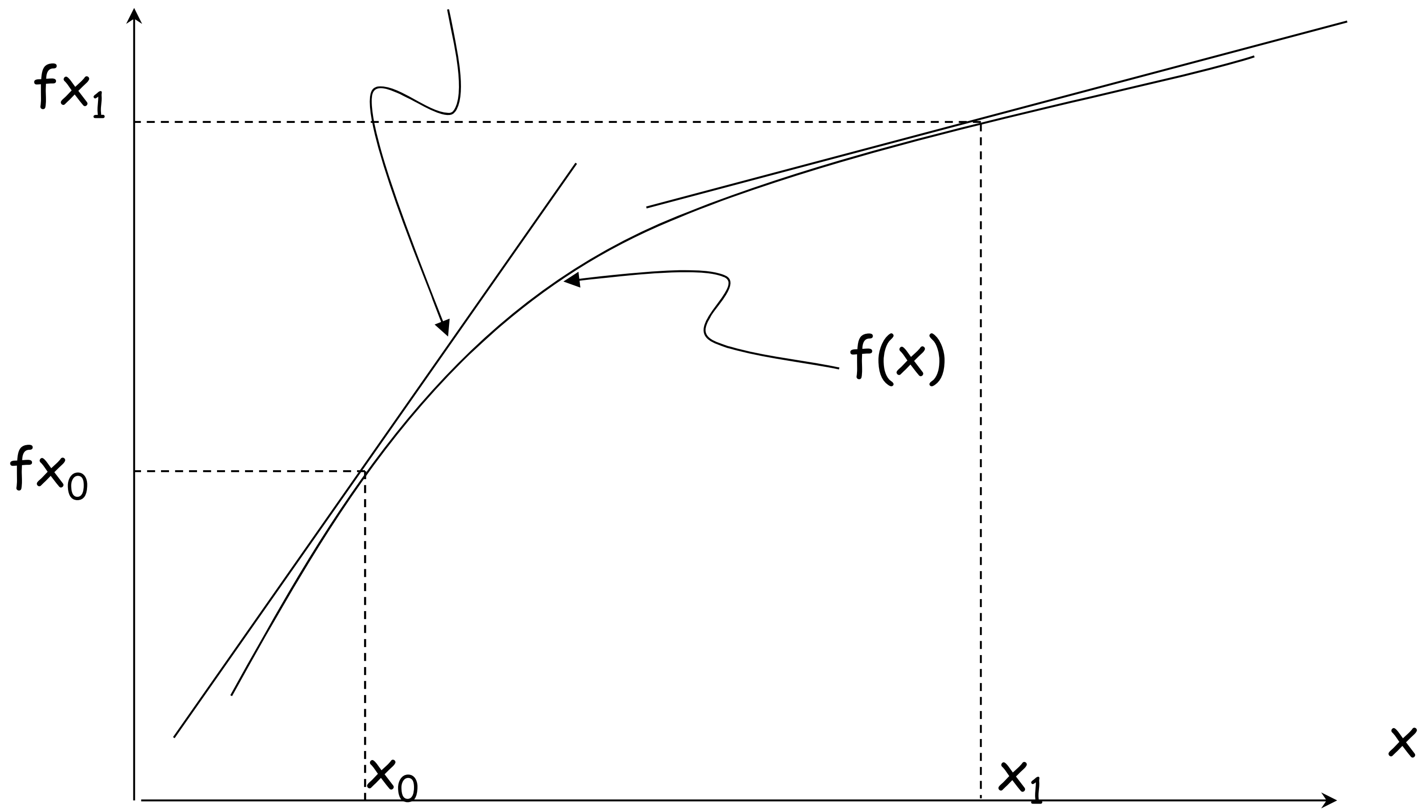
- Linearization consists of truncating the series after the first partial derivatives.

$$f_{(x_1,x_2)} = f_{(\overline{x_1},\overline{x_2})} + \left(\frac{\partial f}{\partial x_1}\right)_{(\overline{x_1},\overline{x_2})}\left(x_1 - \overline{x_1}\right) + \left(\frac{\partial f}{\partial x_2}\right)_{(\overline{x_1},\overline{x_2})}\left(x_2 - \overline{x_2}\right)$$

- We are approximating the real function by a linear function.

Linearization

$$f(x) \cong f(x_0) + (df/dx)_{x0}(x-x_0)$$



References:

- Seborg, D. E., Edgar, T. F., Mellinchamp, D. A. (2003). *Process Dynamics and Control*, 2nd. Edition. John Wiley, ISBN: 978-04-71000-77-8.
- Marlin, T. E. (2000). *Process Control: Designing Processes and Control System for Dynamic Performance*, 2nd. Edition. McGraw Hill, ISBN: 978-00-70393-62-2.
- Stephanopoulos, G. (1984). *Chemical Process Control. An Introduction to Theory and Practice*. Prentice Hall, ISBN: 978-01-31286-29-0