SKF 3143

Process Control and Dynamics:

Laplace Transform

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Learning Objectives

When I complete this chapter, I want to be able to do the following:

1. Solve linear differential equation using Laplace Transform

Introduction

- In previous chapter, we developed a number of mathematical models that describe the dynamic operation of selected processes.
- Solving such models, requires either analytical or numerical integration of the differential equations.
- One important class of models includes systems described by linear differential equations.
- Such linear systems represent the starting point for many analytical techniques in process control.
- In this chapter, we introduce a mathematical tool, the *Laplace transform*, that can significantly reduce the effort required to solve linear differential equation models analytically.
- A major benefit is that this transformation converts differential equations to algebraic equations, which can simplify the mathematical manipulations required to obtain a solution.

The Laplace Transform of Representative Functions

- The Laplace transform of a function *f(t)* is defined as

$$F(s)=L[f(t)]=\int_0^\infty f(t)e^{-st}dt$$

  where *F(s)* is the symbol for the Laplace transform, *f(t)* is some function of time, and *L* is an operator, defined by the integral.
- The Function *f(t)* must satisfy mild conditions which include being piecewise continuous for 0<t<∞.
- When the integration is performed, the transform becomes a function of the Laplace transform variable *s* which is a complex variable.
- The *inverse Laplace transform* (L^{-1}) operates on the function *F(s)* and converts it to *f(t)*.
- One of the important properties of the Laplace transform and the inverse Laplace transform is that they are linear operators.
- The Laplace transformation is a linear operation:

$$L[a_1 f_1(t)+a_2 f_2(t)]=a_1L[f_1(t)]+a_2L[f_2(t)]$$

Laplace Transforms of Some Basic Functions

- **Constant Function.** For *f(t) = a* (a constant),

$$L(a)=\int_0^\infty ae^{-st}dt=-\frac{a}{s}e^{-st}\Big|_0^\infty \quad =0-\left(-\frac{a}{s}\right)=\frac{a}{s}$$

- **Step Function.** The unit step function, defined as

$$S(t)=\begin{cases}0 & t<0\\ 1 & t\geq 0\end{cases}$$

The Laplace transform of the unit step function is similar for the constant *a* above

$$L[S(t)]=\frac{1}{s}$$

Laplace Transforms of Some Basic Functions

- **Derivatives.** The transform of a first derivative is important because such derivatives appear in linear differential equations. This transform is

$$L\left(\frac{df}{dt}\right)=\int_0^\infty \left(\frac{df}{dt}\right)e^{-st}\,dt$$

Integrating by parts,

$$L\left(\frac{df}{dt}\right)=\int_0^\infty f(t)e^{-st}s\,dt+fe^{-st}\Big|_0^\infty$$

$$sL(f)-f(0)=sF(s)-f(0)$$

An *n*th-order derivative, when transformed, yields a series of (n+1) terms:

$$L\left(\frac{d^n f}{dt^n}\right)=s^nF(s)-s^{n-1}f(0)-s^{n-2}f^{(1)}(1)-\ldots-sf^{(n-2)}(0)-f^{(n-1)}(0)$$

Laplace Transforms of Some Basic Functions

- **Exponential Functions.** For a negative exponential, e^{-bt}, with b>0

$$L\left(e^{-bt}\right)=\int_0^\infty e^{-bt}e^{-st}dt=\int_0^\infty e^{-(b+s)t}dt$$

$$=\frac{1}{b+s}\left[-e^{-(b+s)t}\right]_0^\infty=\frac{1}{s+b}$$

- **Trigonometric Functions.** The Laplace transform of *cos* ωt can be calculated using integrating parts.

$$\cos\omega t=\frac{e^{j\omega t}+e^{-j\omega t}}{2}=L(\cos\omega t)=\frac{1}{2}L\left(e^{j\omega t}\right)+\frac{1}{2}L\left(e^{-j\omega t}\right)$$

$$=\frac{1}{2}\left(\frac{1}{s-j\omega}+\frac{1}{s+j\omega}\right)=\frac{1}{2}\left(\frac{s+j\omega+s-j\omega}{s^2+\omega^2}\right)=\frac{s}{s^2+\omega^2}$$

References:

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