SEL4223 Digital Signal Processing

Discrete Fourier Transform

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Introduction

- In digital system, all values must be in discrete.
- However, discrete signal in time-domain that undergoes DTFT will produce a continuous frequency value in frequency-domain.
- Thus, it is impossible to compute Fourier Transform in a digital system.

$$\begin{matrix}\text{Time Domain} \\ \text{n} - \text{discrete}\end{matrix} \quad \xRightarrow{\text{FT}} \quad \begin{matrix}\text{Frequency Domain} \\ \text{f} - \text{continuous}\end{matrix}$$

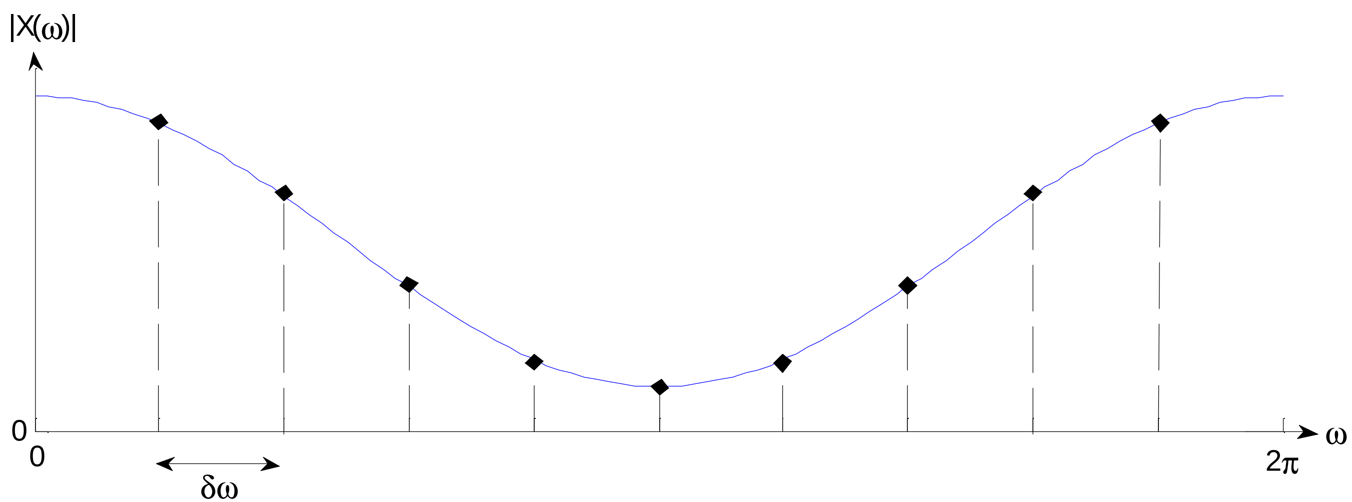
Introduction (cont.)

- To overcome the problem, Discrete Fourier Transform (DFT) is introduced where both time and frequency are in discrete form.

$$\begin{matrix} \text{Time Domain} \\ \text{n} - \text{discrete} \end{matrix} \xRightarrow{\text{DFT}} \begin{matrix} \text{Frequency Domain} \\ \text{k} - \text{discrete} \end{matrix}$$

Frequency Domain Sampling

- In time-domain, sampling is based on T_s (sampling time), where $t = nT_s$. This means that every period of T_s, one sample is taken.
- In frequency-domain, similar process is used to sample the frequency-domain. As shown in figure below, one sample is taken at each $\delta\omega$



Frequency Domain Sampling (cont.)

- $\delta\omega \rightarrow$ equally spaced frequency within the range of $0 \leq \omega \leq 2\pi$

$$\delta\omega = \frac{2\pi}{N}$$

- Where N is the number of samples taken from the frequency domain.
- If in time-domain, $t = nT_s$ is use to sample the time, in frequency-domain, $\omega = k\delta\omega$ is use to sample the frequency where n & k is an integer number.

Frequency Domain Sampling (cont.)

- In time-domain, F_s must be chosen as $F_s \geq 2F_N$ to avoid aliasing.
- In frequency-domain, aliasing can be avoided when N is chosen as $N \geq$ length of signal in the time-domain
- In computing DFT, the frequency range will always start at $\omega = 0$ and end at $\omega = 2\pi - \delta\omega$

DFT Transform Pair

$$X[\mathrm{k}] = \sum_{n=-\infty}^{\infty} x[n]e^{\frac{-j2\pi kn}{N}} \rightarrow forward$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{\frac{j2\pi kn}{N}} \rightarrow inverse$$

Note that the inverse transform of DFT is using summation operation while inverse transform of DTFT uses integration operation.

Example 1

- $h[n] = [1\ 1\ 1\ 1]$

Solution:

- Because the length of the signal is 4, we choose $N = 4$ if other value of N is not stated. Thus, DFT is computed for $0 \leq k \leq 3$ as below where $k = 3$ is at $\omega = 2\pi - \delta\omega = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}$

- $H[k] = \sum_{n=0}^{3} h[n] e^{-\frac{j2\pi kn}{4}} = \sum_{n=0}^{3} h[n] e^{-\frac{j\pi kn}{2}}$
- $H[0] = \sum_{n=0}^{3} h[n] e^{0} = 1 + 1 + 1 + 1 = 4$

Example 1 (cont.)

- $H[1] = \sum_{n=0}^{3} h[n] e^{-\frac{j\pi n}{2}} = 1 + e^{-\frac{j\pi}{2}} + e^{-j\pi} + e^{-\frac{j3}{2}\pi}$

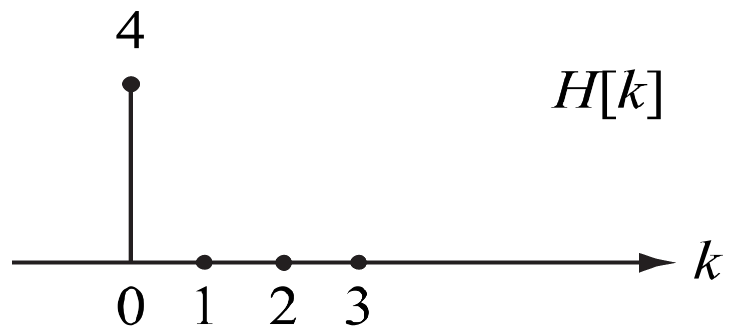
  $= 1 - j - 1 + j = 0$

- $H[2] = \sum_{n=0}^{3} h[n] e^{-j\pi n} = 1 + e^{-j\pi} + e^{-j2\pi} + e^{-j3\pi} = 0$

- $H[3] = \sum_{n=0}^{3} h[n] e^{-\frac{j3\pi n}{2}} = 1 + e^{-\frac{j3\pi}{2}} + e^{-j3\pi} + e^{-\frac{j9}{2}\pi} = 0$

\# $H[z] = [4\ 0\ 0\ 0]$

Example 1 (cont.)



- Based on the example, 4 samples are enough to represent the frequency response that ranges from $\omega = 0$ to $\omega = 2\pi$
- In the example, the samples are taken at each $\delta\omega = \frac{2\pi}{4} = \frac{\pi}{2}$ rad.
- Thus, the samples are actually taken at $\omega = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Example 1 (cont.)

- In this example, $N = 4$ is similar to the signal length. Thus, it is enough to represent the frequency response of the signal and the original signal can be accurately reconstructed from the $H[k]$ where

- $h[n] = \frac{1}{N}\sum_{k=0}^{N-1} H[k]e^{\frac{j2\pi kn}{N}}$
- $h[n] = \frac{1}{4}\sum_{k=0}^{3} H[k]e^{\frac{j\pi kn}{2}}$
- $h[0] = \frac{1}{4}\sum_{k=0}^{3} H[k] = 1$
- $h[1] = \frac{1}{4}\sum_{k=0}^{3} H[k]e^{\frac{j\pi k}{2}} = 1$
- $h[2] = \frac{1}{4}\sum_{k=0}^{3} H[k]e^{j\pi k} = 1$
- $h[3] = \frac{1}{4}\sum_{k=0}^{3} H[k]e^{\frac{j3\pi k}{2}} = 1$

\# $h[n] = [1\ 1\ 1\ 1]$

Example 2

- If $h[n] = x[n] = [1\ 1]$, find $y[n]$ using frequency domain computation

Case 1: choose $\boldsymbol{N = 2}$

- $H[k] = X[k] = \sum_{n=0}^{1} h[n] e^{-j\pi kn}$
- $H[0] = \sum_{n=0}^{1} h[n] = 2$
- $H[1] = \sum_{n=0}^{1} h[n] e^{-j\pi n}$

  $= 1 - 1 = 0$

- Thus,
- $H[k] = X[k] = [2\ \ 0]$

\# $Y[k] = H[k] \cdot X[k] = [4\ \ 0]$

Example 2 (cont.)

- Use IDFT to get back $y[n]$
- $y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k]e^{\frac{j2\pi kn}{N}}$

  $= \frac{1}{2}\sum_{k=0}^{1} Y[k]e^{j\pi kn}$

- $y[0] = \frac{1}{2}(4+0) = 2$
- $y[1] = \frac{1}{2}\sum_{k=0}^{1} Y[k]e^{j\pi k} = 2$

\# $y[n] = \underset{\uparrow}{[2}\ \ 2]$

- The result is wrong because the it is not similar with the time-domain convolution where $y[n] = [1\ \ \underset{\uparrow}{2}\ \ 1]$
- The incorrect answer is due to the insufficient N (sample) used in the DFT computation. As the output $y[n]$ has a length equals to 3, N for DFT should be at least equals to 3.

Example 2 (cont.)

Case 2: choose $\boldsymbol{N = 3}$

- $H[k] = X[k] = \sum_{n=0}^{1} h[n]e^{-\frac{j2\pi kn}{3}}$
- $H[0] = \sum_{n=0}^{1} h[n] = 2$
- $H[1] = \sum_{n=0}^{1} h[n]e^{-\frac{j2\pi n}{3}} = 1 - 0.5 - j0.866 = 0.5 - j0.866 = e^{-j60^o}$
- $H[2] = \sum_{n=0}^{1} h[n]e^{-\frac{j4\pi n}{3}} = 1 - 0.5 + j0.866 = 0.5 + j0.866 = e^{j60^o}$
- $H[k] = [2 \;\; e^{-j60^o} \;\; e^{j60^o}]$

\# $Y[k] = X[k]H[k] = H[k]H[k] = [4 \;\; e^{-j120^o} \;\; e^{j120^o}]$

Example 2 (cont.)

- Then, the inverse transform is
- $y[n] = \frac{1}{3}\sum_{k=0}^{2} Y[k]e^{j\frac{2}{3}\pi kn}$
- $y[0] = \frac{1}{3}\sum_{k=0}^{2} Y[k] = \frac{1}{3}\left(4 + e^{-j(120^o)} + e^{j(120^o)}\right)$

$$= \frac{1}{3}(4 + 2\cos(120^o)) = 1$$

- $y[1] = \frac{1}{3}\sum_{k=0}^{2} Y[k]e^{j\frac{2}{3}\pi k}$

$$= \frac{1}{3}\left(4 + e^{j(-120^o+120^o)} + e^{j(120^o+240^o)}\right)$$

$$= \frac{1}{3}(4 + 1 + 1) = 2$$

Example 2 (cont.)

- $y[2] = \frac{1}{3}\sum_{k=0}^{2} Y[k]e^{j\frac{4}{3}\pi k}$

$$= \frac{1}{3}\left(4 + e^{j(-120^o+240^o)} + e^{j(120^o+480^o)}\right)$$

$$= \frac{1}{3}(4 + 2\cos(120^o)) = 1$$

$y[n] = [1 \;\; 2 \;\; 1]$

- This result is similar when computed based on the time-domain convolution

Example 2 (cont.)

- Find $Y[k]$ if $x[n] = [1\ 1]$ and $h[n] = [1\ 1\ 1]$

Solution:

- In order to compute $Y[k]$, DFT for both $x[n]$ and $h[n]$ must be first computed where both DFT computation must use similar N. This is to ensure that multiplication of $X[k]$ and $H[k]$ is made on each similar ω value.

- Let see what happen if different N value is chosen, for example $N = 2\ for\ x[n]$ and $N = 3\ for\ h[n]$.
  - For $N = 2$, $\delta\omega = \pi$, thus $k = [0, 1]$ is referring to $\omega = [0, \pi]$
  - For $N = 3$, $\delta\omega = \frac{2\pi}{3}$, thus $k = [0, 1, 2]$ is referring to $\omega = \left[0, \frac{2\pi}{3}, \frac{4\pi}{3}\right]$

Example 2 (cont.)

- For different N, it can be seen that even at similar k values, the ω value is different. Thus, multiplication of $X[k]$ and $H[k]$ will be wrong.
- To get a correct answer, N must be set at least with its minimum value which is 4. Thus, both DFT of $x[n]$ and $h[n]$ must be computed based on 4-points DFT. Below are the results
- $X[k] = [2, \quad 1 - j, \quad 0, \quad 1 + j]$
- $H[k] = [3, \quad -j, \quad 1, \quad j]$
- $Y[k] = [6, -1 - j, 0, -1 + j]$

\# $y[n] = [1\ 2\ 2\ 1]$

Relationship of DFT to DTFS

- It is known that, frequency in DTFS is also discrete such in the DFT. Does DFT and DTFT similar?

- Basically, difference of the two is DTFS is for periodic signal while DFT is for aperiodic signal as it is derived from DTFT. However, let's look at their formulation as below. N in DTFS refers to the number of samples in one time-period while N in DFT refers to the number of samples in the frequency domain.

$$DTFS \quad \Longrightarrow \quad a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x_{periodic}[n] e^{-j2\pi kn/N}$$

$$DFT \quad \Longrightarrow \quad X[\mathrm{k}] = \sum_{n=-\infty}^{\infty} x_{aperiodic}[n] e^{-j2\pi kn/N}$$

Relationship of DFT to DTFS (cont.)

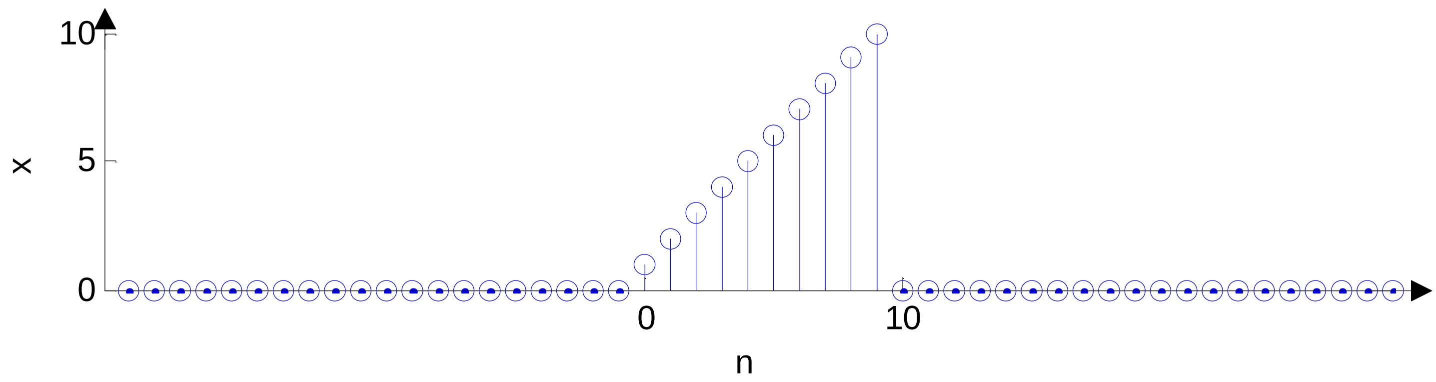
- If the length of the aperiodic signal is equals to the number of sample in one period of the periodic signal as shown in the next two figure and frequency in DFT is sampled at 10 point, the DTFS and the DFT can be computed as below:

$$DTFS \quad \Longrightarrow \quad a_k = \frac{1}{10}\sum_{n=0}^{9} x[n]e^{-j2\pi kn/10}$$

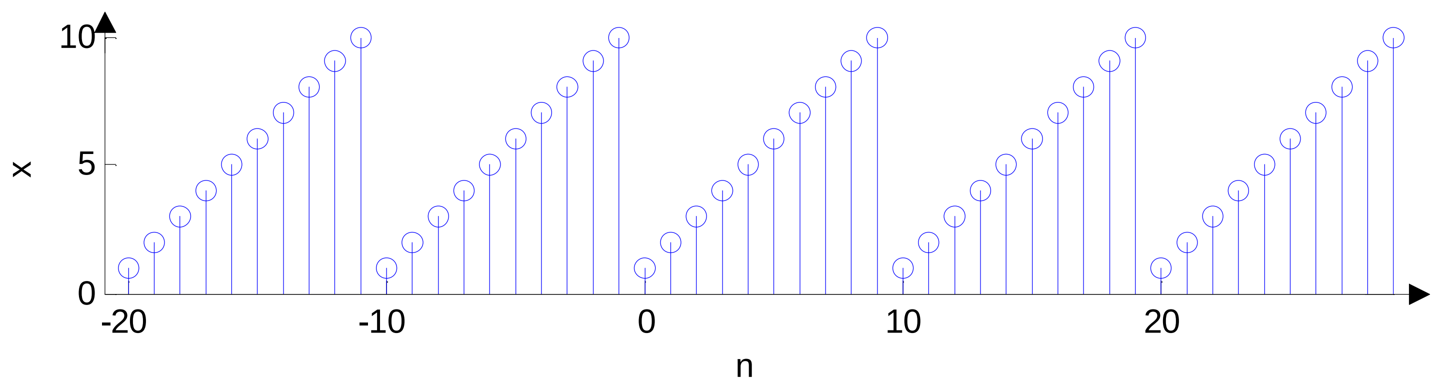
$$DFT \quad \Longrightarrow \quad X[\mathrm{k}] = \sum_{n=0}^{9} x[n]e^{-j2\pi kn/10}$$

- From there, it can be seen that computation of DTFS and DFT identical except that DTFS is divided by 10. This phenomena cause the operation of DFT to be influence by the characteristic of the periodic signal, which is called 'circular effect'.

Relationship of DFT to DTFS (cont.)



Aperiodic signal with length 10 samples



Periodic signal with 10 samples in each period

DFT Properties

- Properties of the DFT is based on the ‘circular effect’

Properties	Time Domain	Frequency Domain
Notation	$x[n], y[n]$	$X[k], Y[k]$
Periodicity	$x[n] = x[n+N]$	$X[k] = X[k+N]$
Linearity	$ax[n] + by[n]$	$aX[k] + bY[k]$
Circular time shifting	$x([n-n_d])_N$	$e^{-j2\pi k n_d/N}X[k]$
Circular frequency shifting	$e^{j2\pi k_d n/N}x[n]$	$X([k-k_d])_N$
Circular convolution	$x[n] \, N \, y[n]$	$X[k]Y[k]$
Multiplication	$x[n]y[n]$	$\frac{1}{N}X[k] \, N \, Y[k]$
Parseval’s theorem	$\sum_{n=0}^{N-1} x[n]y^*[n]$	$\frac{1}{N}\sum_{k=0}^{N-1} X[k]Y^*[k]$

Example 4

- Let's get back to Example 2 where $h[n] = x[n] = [1\ 1]$ and $N = 2$.
- Applying DFT to both signal results $H[k] = X[k] = [2\ 2]$
- Thus, $Y[k] = X[k]H[k] = [4\ 4]$
- Then $y[n]$ is obtain by taking the inverse DFT of $Y[k]$ where $y[n] = [2\ 2]$.
- The results is different compare to the normal time-domain convolution operation where $y[n] = [1\ 2\ 1]$.
- This is because DFT multiplication in frequency domain does not results normal convolution in time-domain. Instead, it results circular convolution as shown in the next figure where the aperiodic signals of $x[n]$ and $h[n]$ act as periodic signal.

Example 4 (cont.)

$h[n] \Rightarrow$	1	1	**1**	**1**	1	1	1	1	1
$x[n] \Rightarrow$	1	1	1	**1**	**1**	1	1	1	1

- The term circular comes in because operation to periodic signal within N samples window can be observed as circulating the values inside the window
- Example 5 shows an example.

Example 5

- Do circular convolution to $x_1[n] = [1\ 2\ 3]$ and $x_2[n] = [4\ 5\ 6]$. Let the output as $y[n]$

$$n = 0, \quad \begin{array}{r} x_1[-k] \Rightarrow \\ x_2[k] \Rightarrow \end{array} \begin{array}{ccc} [1 & 3 & 2] \\ [4 & 5 & 6] \end{array} \Rightarrow y[0] = 31$$

$$n = 1, \quad \begin{array}{r} x_1[1-k] \Rightarrow \\ x_2[k] \Rightarrow \end{array} \begin{array}{ccc} [2 & 1 & 3] \\ [4 & 5 & 6] \end{array} \Rightarrow y[0] = 31$$

$$n = 2, \quad \begin{array}{r} x_1[2-k] \Rightarrow \\ x_2[k] \Rightarrow \end{array} \begin{array}{ccc} [3 & 2 & 1] \\ [4 & 5 & 6] \end{array} \Rightarrow y[0] = 28$$

- Thus, $y[n] = [31\ 31\ 28]$

Example 5 (cont.)

- Repeat the circular convolution with $x_1[n] = [1\ 2\ 3\ 0\ 0]$ and $x_2[n] = [4\ 5\ 6\ 0\ 0]$
- The result is $y[n] = [4\ \ 13\ \ 28\ \ 27\ \ 18]$, which is different compare to the previous answer but similar with normal convolution output.
- In Example 1, it was shown that insufficient N samples gives different output compare to the normal convolution process. From Example 5, it is known that the different is because of the circular effect.
- To avoid circular effect,

$$N \geq length\ x[n] + length\ h[n] - 2$$

Symmetry Properties of DFT

- In general, the symmetry property can be written as

$$H[k]^* = H[N-k]$$

- Thus, to obtain $H[k]$, only values at $\omega \leq \pi$ need to be computed.

Example 6

- $h[n] = [1\ 2\ 2\ 2\ 1]$

$\uparrow$

$H[k] = \sum_{n=0}^{4} h[n] e^{-\frac{j2\pi kn}{5}}$

- k values that are at $\omega \leq \pi$ is $k = 0{,}1\ \&\ 2$

$H[0] = \sum_{n=0}^{4} h[n] = 8$

$H[1] = \sum_{n=0}^{4} h[n] e^{-\frac{j2\pi n}{5}} = -1.309 - j0.951$

$H[2] = \sum_{n=0}^{4} h[n] e^{-\frac{j4\pi n}{5}} = -0.191 - j0.588$

Example 6 (cont.)

- From there

$$H[3] = H[2]^* = -0.191 + j0.588$$

$$H[4] = H[1]^* = -1.309 + j0.951$$

- When magnitude and phase spectrum are computed, the results are

$$|H[k]| = [8 \quad 1.618 \quad 0.618 \quad 0.618 \quad 1.618]$$

$$\angle H(\omega) = [0 \quad -2.5133 \quad -1.8849 \quad 1.8849 \quad 2.5133]$$

References

1) John G. Proakis, Dimitris K Manolakis, “Digital Signal Processing: Principle, Algorithm and Applications”, Prentice-Hall, 4th edition (2006).

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3) Alan V. Oppenheim, Ronald W. Schafer, “Discrete-Time Signal Processing”, Prentice-Hall, 3rd edition (2009).