SEL4223 Digital Signal Processing

Inverse Z-Transform

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Inverse Z-Transform

- Transform from z-domain to time-domain

$$x[n] = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz$$

- Note that the mathematical operation for the inverse z-transform use circular integration instead of summation. This is due to the continuous value of the z.

Z-Transform pair Table

- The inverse z-transform equation is complicated. The easier way is to use the z-transform pair table

Time-domain signal	z-transform	ROC
1) $\delta[n]$	1	All z
2) $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3) $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4) $\delta[n-m]$	z^{-m}	$z \neq 0$ if $m > 0$ $z \neq \infty$ if $m < 0$
5) $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$z > a$
6) $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$z < a$

Time-domain signal	z-transform	ROC
7) $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$z > a$
8) $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$z < a$
9) $\cos(\omega_o n)u[n]$	$\frac{1-cos(\omega_o)\,z^{-1}}{1-2\,cos(\omega_o)z^{-1}+z^{-2}}$	$\|z\| > 1$
10) $\sin(\omega_o n)\,u[n]$	$\frac{1-sin(\omega_o)z^{-1}}{1-2\,cos(\omega_o)z^{-1}+z^{-2}}$	$\|z\| > 1$
11) $r^n\cos(\omega_o n)u[n]$	$\frac{1-rcos(\omega_o)\,z^{-1}}{1-2\,rcos(\omega_o)z^{-1}+r^2z^{-2}}$	$\|z\| > \|r\|$
12) $r^n\sin(\omega_o n)u[n]$	$\frac{1-rsin(\omega_o)z^{-1}}{1-2\,rcos(\omega_o)z^{-1}+r^2z^{-2}}$	$\|z\| > \|r\|$
13) $\begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$	$\frac{a^N z^{-N}}{z^{(-1)}}$	$\|z\| > 0$

Example 1

- $H(z) = \frac{3}{1+\frac{3}{4}z^{-1}}$ $\qquad ROC, |z| > \frac{3}{4}$
- From the table, we can use the z-transform pair no 5.
- $a^n u[n] \overset{z}{\Longleftrightarrow} \frac{1}{(1-az^{-1})}$, ROC, $|z| > |a|$
- Thus, $H(z) = \frac{3}{1+\frac{3}{4}z^{-1}} = 3\left(\frac{1}{1-\left(-\frac{3}{4}\right)z^{-1}}\right)$

$$h[n] = 3\left(-\frac{3}{4}\right)^n u[n]$$

Example 2

- $H(z) = \frac{3}{1+\frac{3}{4}z^{-1}}, \qquad ROC, |z| < \frac{3}{4}$
- Use pair no. 6

$$-a^n u[-n-1] \overset{z}{\Longleftrightarrow} \frac{1}{1-az^{-1}}, \qquad ROC, |z| < |a|$$

- Thus,

$$h[n] = -3\left(-\frac{3}{4}\right)^n u[-n-1]$$

Example 3

- $H(z) = 1 + z^{-1} + z^{-3}$, ROC, $|z| > 0$
- User pair no. 4

$$\delta[n-m] \overset{z}{\Longleftrightarrow} z^{-m}$$

- Thus,

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-3]$$

Partial Fraction Expansion

- $H(z)$ can also has a different form than all the listed pairs in the z-transform table. Thus, $H(z)$ needs to be rearranged to become alike with one of the forms listed in the table. This can be done by performing partial fraction expansion.

- In general, we can write $H(z)$ as

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= \frac{\sum_{k=0}^{M} b_k \, z^{-k}}{\sum_{k=0}^{N} a_k \, z^{-k}}$$

Partial Fraction Expansion (cont.)

- Based on the equation above, three cases will be discussed

  1) $\mathrm{N} > \mathrm{M}$ and all poles are different

  2) $\mathrm{M} \geq \mathrm{N}$ and all poles are different

  3) More than 1 poles are similar

$N > M$ and all poles are different

$$H(z) = \frac{\sum_{k=0}^{M} b_k\, z^{-k}}{\sum_{k=0}^{N} a_k\, z^{-k}} = \frac{\sum_{k=0}^{M} b_k\, z^{-k}}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

$$= \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

Where $A_k = (1 - d_k z^{-1})H(z)|_{z=d_k}$

Example 4

- Find $h[n]$ where

$$H(z) = \frac{(1-2z^{-1})}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}, \qquad |z| > \frac{1}{2}$$

Solution:

- $$H(z) = \frac{1-2z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} = \sum_{k=1}^{2} \frac{A_k}{1-d_k z^{-1}}$$

$$= \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-\frac{1}{3}z^{-1}}$$

Example 4 (cont.)

- $A_1 = \left(1 - \frac{1}{2}z^{-1}\right)H(z)\Big|_{z=\frac{1}{2}}$

$= \frac{1-2z^{-1}}{1-\frac{1}{3}z^{-1}}\Big|_{z=\frac{1}{2}}$

$= \frac{1-2(2)}{1-\frac{1}{3}(2)}$

$= -9$

- $A_2 = \left(1 - \frac{1}{3}z^{-1}\right)H(z)\Big|_{z=\frac{1}{3}}$

$= \frac{1-2z^{-1}}{1-\frac{1}{2}z^{-1}}\Big|_{z=\frac{1}{3}}$

$= \frac{1-2(3)}{1-\frac{1}{2}(3)}$

$= 10$

Example 4 (cont.)

- Thus,

$$H(z) = -\frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{10}{1 - \frac{1}{3}z^{-1}}$$

- From table, use pair no. 5 to get $h[n]$

$$h[n] = -9\left(\frac{1}{2}\right)^n u[n] + 10\left(\frac{1}{3}\right)^n u[n]$$

$M \geq N$ and all poles are at different

1) Do long division until $N > M$

2) For remainder of the long division, use procedure for $N > M$

Example 5

- Find $h[n]$ where

$$H(z) = \frac{1+2z^{-1}-5z^{-2}+6z^{-3}}{1-3z^{-1}+2z^{-2}}, \quad |z| > 2$$

Solution:

- Because $M > N$, do the long division

$$
\begin{array}{r|l}
 & 2 + 3z^{-1} \\
\hline
1 - 3z^{-1} + 2z^{-2} & 1 + 2z^{-1} - 5z^{-2} + 6z^{-3} \\
 & \quad 3z^{-1} - 9z^{-2} + 6z^{-3} \\
\hline
 & 1 - z^{-1} + 4z^{-2} \\
 & 2 - 6z^{-1} + 4z^{-2} \\
\hline
 & -1 + 5z^{-1}
\end{array}
$$

Example 5 (cont.)

- Thus, $H(z) = 2 + 3z^{-1} + \frac{-1+5z^{-1}}{1-3z^{-1}+2z^{-2}}$
- The first two expressions are results of the long division while the third is the remainder of the long division
- For the remainder, use procedure for $N > M$
- $H_r(z) = \frac{-1+5z^{-1}}{1-3z^{-1}+2z^{-2}}$

$$= \frac{(-1+5z^{-1})}{(1-z^{-1})(1-2z^{-1})}$$

$$= \frac{A_1}{1-z^{-1}} + \frac{A_2}{1-2z^{-1}}$$

Example 5 (cont.)

- $A_1 = (1 - z^{-1})H_r(z)|_{z=1}$

  $= \frac{-1+5z^{-1}}{1-2z^{-1}}\Big|_{z=1}$

  $= \frac{-1+5}{1-2} = -4$

- $A_2 = (1 - 2z^{-1})H_r(z)|_{z=2}$

  $= \frac{-1+5z^{-1}}{1-z^{-1}}\Big|_{z=2}$

  $= \frac{-1+\frac{5}{2}}{1-\frac{1}{2}} = 3$

- $H_r(z) = -\frac{4}{1-z^{-1}} + \frac{3}{1-2z^{-1}}$
- $H(z) = 2 + 3z^{-1} - \frac{4}{1-z^{-1}} + \frac{3}{1-2z^{-1}}$
- Use pair 4 & 5; $h[n] = 2\delta[n] + 3\delta[n-1] - 4u[n] + 3(2)^n u[n]$

More than 1 poles are similar

- In general, the z-transform expression can be written as

$$H(z) = \underbrace{\sum_{r=1}^{M-N} B_r z^{-r}}_{①} + \underbrace{\sum_{k=1.k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}}}_{②} + \underbrace{\sum_{m=1}^{S} \frac{C_m}{(1 - d_i z^{-1})^m}}_{③}$$

If $N > M$ and all poles are different, only (2) exists

If $M \geq N$ and all poles are different, only (1) and (2) exist

If $N > M$ and more than 1 poles are similar, only (2) and (3) exist

If $M \geq N$ and more than 1 poles are similar, (1), (2) and (3) exist

More than 1 poles are similar

- Thus, when more than 1 poles are similar, expression (3) exists where

- $$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \cdot \left\{ \frac{d^{s-m}}{dz^{-(s-m)}} \left[(1 - d_i z^{-1})^s H(z) \right] \right\} \Bigg|_{z=d_i}$$

  $for \;\; m \neq s$

- $C_s = (1 - d_i z^{-1})^s H(z)|_{z=d_i}$

Example 6

- Find $h[n]$ where

$$H(z) = \frac{z^{-1}}{(1 - z^{-1})\left(1 - \frac{1}{2}z^{-1}\right)^2}$$

- In this example, there are 3 poles where 2 of the poles are similar. The poles are, $z = 1, z = \frac{1}{2}$ & $z = \frac{1}{2}$

- Because $N > M$, no need for long division operation

Example 6 (cont.)

- $H(z) = \frac{z^{-1}}{(1-z^{-1})\left(1-\frac{1}{2}z^{-1}\right)^2}, \qquad |z| > 1$

$$= \frac{A_1}{1-z^{-1}} + \frac{C_1}{1-\frac{1}{2}z^{-1}} + \frac{C_2}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

- $A_1 = (1 - z^{-1})H(z)|_{z=1}$

$$= \left.\frac{z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}\right|_{z=1}$$

$$= \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4$$

Example 6 (cont.)

- $$C_1 = \frac{1}{(2-1)!\left(-\frac{1}{2}\right)^{2-1}}\left\{\frac{d}{dz^{-1}}\left[\left(1-\frac{1}{2}z^{-1}\right)^2 H(z)\right]\right\}\Bigg|_{z=\frac{1}{2}}$$

$$= -2\left\{\frac{d}{dz^{-1}}\frac{z^{-1}}{1-z^{-1}}\right\}\Bigg|_{z=\frac{1}{2}}$$

$$= -2\left(\frac{-1+2}{(-1)^2}\right)$$

$$= -2$$

Example 6 (cont.)

- $C_2 = C_s = \left(1 - \frac{1}{2} z^{-1}\right)^2 H(z)\Big|_{z=\frac{1}{2}}$

$$= \frac{z^{-1}}{1 - z^{-1}}\Big|_{z=\frac{1}{2}}$$

$$= \frac{2}{1-2}$$

$$= -2$$

Example 6 (cont.)

- Thus,

$$H(z) = \frac{4}{1 - z^{-1}} - \frac{2}{1 - \frac{1}{2}z^{-1}} - \frac{2}{\left(1 - \frac{1}{2}z^{-1}\right)^2}$$

- From table,

$$h[n] = 4u[n] - 2\left(\frac{1}{2}\right)^n u[n] - 4(n+1)\left(\frac{1}{2}\right)^{n+1} u[n+1]$$

- Try to figure out which table were used for the inverse

Z-Transform Properties

Property	$h[n]$	$H(\omega)$	ROC
Linearity	$ah_1[n] + bh_{2[n]}$	$aH_1(z) + bH_2(z)$	$ROC_{H_1} \cap ROC_{H_2}$
Time-shifting	$h[n - n_d]$	$z^{-n_d}H(z)$	That of $H(z)$, except $z = 0$ if $n_d > 0$ and $z = \infty$ if $n_d < 0$
Scaling in the z-domain	$a^n h[n]$	$H\left(\frac{z}{a}\right)$	$\|a\|ROC_{H(z)}$
Time-reversal	$h[-n]$	$H(z^{-1})$	$ROC_{H(z)}^{-1}$
Differentiation	$nh[n]$	$-z\left(\frac{dH(z)}{dz}\right)$	$ROC_{H(z)}$
Conjugation	$h^*[n]$	$H^*[z^*]$	$ROC_{H(z)}$
Convolution	$h_1[n] * h_2[n]$	$H_1(z)H_2(z)$	$ROC_{H_1} \cap ROC_{H_2}$
Initial value theorem	If $h[n]$ causal	$h[0] = \lim_{z\to\infty} H(z)$	-

Example 7: Linearity

$$\alpha h_1[n] + \beta h_2[n] \overset{z}{\Longleftrightarrow} \alpha H_1(z) + \beta H_2(z), \qquad ROC_{H_1} \cap ROC_{H_2}$$

- $h[n] = \left(-\frac{1}{2}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$

Solution:

- $H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{2}{1-\frac{1}{3}z^{-1}}$
- $ROC_1, \quad |z| > \frac{1}{2}, \qquad ROC_2, \quad |z| > \frac{1}{3}$
- Thus, ROC = $ROC_1 \cap ROC_2, \quad |z| > \frac{1}{2}$

Example 8: Linearity

- $h[n] = 3\delta[n-1] + 2\left(\frac{1}{2}\right)^n u[-n-1]$

Solution:

- $H(z) = 3z^{-1} - \frac{2}{1-\frac{1}{2}z^{-1}}$
- $ROC_1, \quad |z| > 0, \qquad ROC_2, \quad |z| < \frac{1}{2}$
- $ROC = ROC_1 \cap ROC_2, \qquad 0 < |z| < \frac{1}{2}$

Example 9: Time Shifting

$$h[n-n_d] \overset{z}{\Longleftrightarrow} z^{-n_d}H(z)$$

- $h[n] = \delta[n-3] + 2^{n-2}u[n-2]$

Solution:

- $H(z) = z^{-3} + \frac{z^{-2}}{1-2z^{-1}}, \qquad |z| > 2$

Example 10: Time Shifting

- $h[n] = \left(\frac{1}{2}\right)^{n-3} u[n-2]$

Solution:

- $h[n] = \left(\frac{1}{2}\right)^{(n-2)-1} u[n-2]$

$$= \left(\frac{1}{2}\right)^{-1} \left(\frac{1}{2}\right)^{n-2} u[n-2]$$

$$= 2\left(\frac{1}{2}\right)^{n-2} u[n-2]$$

- $H(z) = \frac{2z^{-2}}{1-\frac{1}{2}z^{-1}}$

Example 11: Scaling in z-domain

$$a^n h[n] \overset{z}{\Longleftrightarrow} H\left(\frac{z}{a}\right)$$

$$ROC = |a|.\, ROC_H$$

- If $h[n] = u[n]$, find $H_2(z)$ where $h_2[n] = 2^n h[n]$

Solution:

- For $h[n] = u[n]$, its $H(z) = \frac{1}{1-z^{-1}}$
- Thus, $H_2(z) = H\left(\frac{z}{2}\right) = \frac{1}{1-\left(\frac{z}{2}\right)^{-1}} = \frac{1}{1-2z^{-1}}$, $ROC, |z| > 2$

Example 12: Scaling in z-domain

- $h[n] = 2^n(\delta[n] + \delta[n-1] + \delta[n-2])$

Solution:

- 2 methods can be used to solve the problem

(a) Using time shifting and linear properties,

$$h[n] = 2^n\delta[n] + 2^n\delta[n-1] + 2^n\delta[n-2]$$

$$= \delta[n] + 2\delta[n-1] + 4\delta[n-2]$$

$$H(z) = 1 + 2z^{-1} + 4z^{-2}$$

Example 12: Scaling in z-domain (cont.)

(b) Using scaling in z-domain properties

If $h_1[n] = \delta[n] + \delta[n-1] + \delta[n-2]$

$$H_1(z) = 1 + z^{-1} + z^{-2}$$

Thus, $H(z) = H_1\left(\frac{z}{2}\right)$

$$= 1 + \left(\frac{z}{2}\right)^{-1} + \left(\frac{z}{2}\right)^{-2}$$

$$= 1 + 2z^{-1} + 4z^{-2}$$

Example 13: Differentiation

$$nh[n] \overset{z}{\Longleftrightarrow} -z\frac{dH(z)}{dz}$$

- $h[n] = nu[n]$

Solution:

- $u[n] \overset{Z}{\rightarrow} (\frac{1}{1-z^{-1}})$
- $H(z) = -z.\frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right)$

$$= -z.\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= -z.\frac{(z-1)(1)-(z)(1)}{(z-1)^2}$$

$$= \frac{-z^2+z+z^2}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}$$

$$= \frac{z^{-1}}{(1-z^{-1})^2}$$

Conjugation

$$h^*[n] \overset{z}{\Longleftrightarrow} H^*[z^*]$$

- What is conjugate?

  For complex number $a + jb$, its conjugate is $a - jb$. The sign of the imaginary value is changed.

- Complex number can also be written as, $re^{j\theta}$, where $r = \sqrt{a^2 + b^2}$, and $\theta = \tan^{-1}\frac{b}{a}$

- Thus, conjugate for $re^{j\theta}$ is $re^{-j\theta}$

Example 14: Conjugation

- If $h[n] = e^{jn}u[n]$, its $H(z) = \frac{1}{1-e^{j}z^{-1}}$
- Thus, for $h_1[n] = h^*[n]$,
- $H_1(z) = H^*(z^*)$

$$= \frac{1}{1-e^{-j}z^{-1}}$$

Example 15: Time-reversal

$$h[-n] \overset{z}{\Longleftrightarrow} H\left(\frac{1}{z}\right), \qquad ROC = ROC_H^{-1}$$

- $x[n] = a^n u[n]$
- Find, $Y(z)$ if $y[n] = x[-n]$
- **Solution:**
- $X(z) = \frac{1}{1-az^{-1}}, \quad |z| > a$
- Thus,
- $Y(z) = \frac{1}{1-\left(\frac{1}{z}\right)^{-1}}$

$$= \frac{1}{1-z}, \qquad |z| > \frac{1}{a}$$

Example 16: Convolution

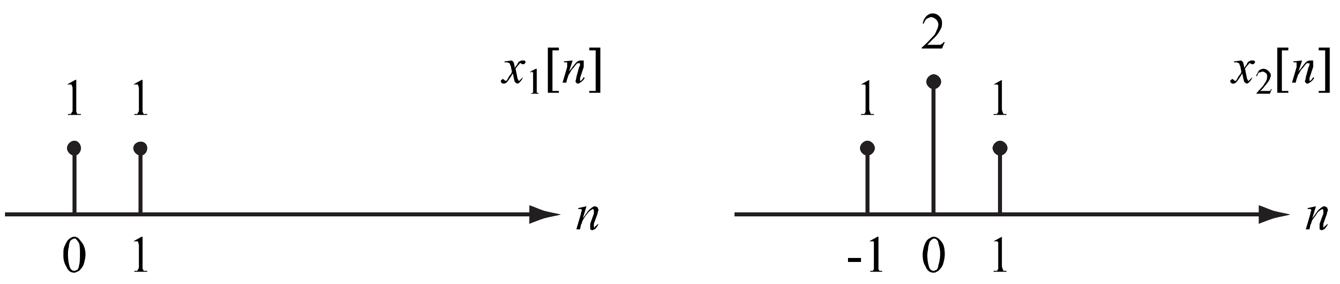
$$x_1[n] * x_2[n] \overset{z}{\Longleftrightarrow} X_1(z)X_2(z)$$

- $x[n] = u[n],$ Find $Y(z)$ if $y[n] = x[n] * x[-n]$

Solution:

- $x[n] \overset{Z}{\rightarrow} \left(\frac{1}{1-z^{-1}}\right),\quad x[-n] \overset{Z}{\rightarrow} \left(\frac{1}{1-z}\right)$
- $Thus,\ \ Y(z) = \frac{1}{1-z^{-1}} . \frac{1}{1-z} = \frac{1}{(1-z^{-1})(1-z)}$

Example 17: Convolution



- $y[n] = x_1[n] * x_2[n]$, Find $Y(z)$

Solution:

- $x_1[n] = \delta[n] + \delta[n-1] \implies X_1(z) = 1 + z^{-1}$
- $x_2[n] = \delta[n+1] + 2\delta[n] + \delta[n-1] \implies X_2(z) = z + 2 + z^{-1}$
- $Y(z) = X_1(z)X_2(z)$

$$= (1 + z^{-1})(z + 2 + z^{-1})$$

$$= z + 3 + 3z^{-1} + z^{-2}$$

Example 18: Initial Value Theorem

For causal signal where $h[n] = 0$ for $n < 0$,
its initial value can be estimated by

$$h[0] = \lim_{z \to \infty} H(z)$$

- Find initial value, $h[0]$ for $h[n] = u[n]$

Solution:

- $H(z) = \frac{1}{1-z^{-1}}$
- $h[0] = \lim_{z \to \infty} \left(\frac{1}{1-z^{-1}} \right) = \frac{1}{1-\infty^{-1}} = 1$

Example 19: Initial Value Theorem

- Find initial value, $h[0]$ for $h[n] = (\cos(n))u[n]$

Solution:

- $H(z) = \frac{1-(\cos(1))z^{-1}}{(1-(2\cos(1))z^{-1}+z^{-2})}$
- $h[0] = \lim_{z\to\infty} H(z)$

$$= \frac{1-(\cos(1))/\infty}{(1-(2\cos(1))/\infty+\infty^{-2})} = \frac{1}{1} = 1$$

References

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